Rigorous || Structured || Proof-Oriented

Refoma Core | Foundations of Mathematics | 2026

ANALYSIS ONE

This book introduces the fundamental concepts of mathematical analysis, including limits, continuity, and differentiation. It adopts a rigorous and structured approach, supported by examples and detailed exercises, to help students develop logical reasoning and a solid analytical foundation.

Abdeljalil Mostefaoui | Echahid Cheikh Larbi Tebessi university

Analysis I

Refoma Core — Analysis

Course Summary & Solved Exercises

Subject: Analysis I
Educational level: 1st Year University mathematics
Student: Abdeljalil Mostefaoui

Math Note:
Analysis reveals that what appears arbitrary is governed by precise laws.

Designed to support students in mastering the fundamental concepts of Analysis.

Abdeljalil Mostefaoui | Echahid Cheikh Larbi Tebessi university

First Edition

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Typeset in LaTeX.

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Introduction

This document presents a comprehensive and rigorous treatment of the fundamental topics in **Analysis I**. Its primary objective is to develop a solid understanding of real analysis through precise definitions, well-structured theorems, and formally written proofs, in accordance with university-level mathematical standards.
The material emphasizes logical reasoning and mathematical rigor, aiming to help students transition from computational mathematics to proof-based analysis.

Methodological Framework:

The approach adopted in this treatise is characterized by a commitment to formal rigor and logical precision. This methodological framework is structured around the following core principles:

- **Explicit Statement of Logical Structure:** For every theorem, lemma, and problem, the underlying hypotheses (premises) and conclusions are unambiguously identified. This practice reinforces the conditional nature of mathematical statements and clarifies the direction of implication in each argument.
- **Definitional Rigor:** All arguments are constructed strictly upon the foundational definitions introduced (e.g., the field and order axioms for $\mathbb{R}$, the precise ϵ–N definition of a sequence limit, the ϵ–δ definition of continuity). No result is assumed without justification from these first principles or previously established theorems.
- **Systematic and Transparent Derivation:** Results are obtained through a systematic, step-by-step logical derivation. Each step is explicitly justified by citing a definition, an axiom, a previously proven result, or a valid rule of inference. Intuitive geometric or heuristic reasoning may be offered for motivation but is never accepted as a substitute for a formal proof.
- **Precise Handling of Quantifiers:** Meticulous attention is paid to the manipulation of logical quantifiers (universal $\forall$ and existential $\exists$). Key proof techniques, such as proving a "for all" statement by considering an arbitrary element, or establishing an "there exists" statement by explicit construction or application of an existence theorem, are employed correctly and their logic is made explicit.
- **Formal Proof Architecture:** Standard proof techniques–including direct proof, proof by contrapositive, proof by contradiction, and mathematical induction–are used consistently. The structure of each proof (e.g., "Assume P. We will show Q...") is clearly delineated to enhance readability and verifiability.
- **Notational Consistency:** A uniform and standard mathematical notation is maintained throughout. This eliminates ambiguity and ensures that symbolic expressions are parsed in a single, universally understood manner within the context of real analysis.

Dedication

All praise and gratitude are due to Allah, the Almighty, whose guidance, mercy, and blessings have made this work possible. Through His wisdom and support, I was granted the strength, patience, and determination required to pursue knowledge and to complete this academic journey.

I dedicate this work with profound respect, love, and gratitude to my parents, who have been the cornerstone of my education. To my beloved father, **Abdelhamid Mostefaoui**, whose unwavering dedication, sacrifices, and moral guidance have always inspired me to strive for excellence. His constant encouragement, wisdom, and belief in the value of education have shaped my character and motivated me to persevere through challenges.

To my dear mother, **Mebarka.W.Z**, whose endless love, patience, and prayers have been my greatest source of strength. Her quiet sacrifices, sincere support, and unconditional care have illuminated my path and provided me with the confidence and serenity necessary to pursue my studies with determination and humility. Words cannot fully express my gratitude for her role in every achievement I reach.

I also extend my sincere appreciation and deep respect to my teacher, **Belkacem Rebiai**, whose seriousness, discipline, and commitment to teaching have left a lasting impact on my academic formation. His dedication, high standards, and tireless efforts to guide and support us have not only enriched our knowledge but also instilled in us the values of rigor, responsibility, and intellectual honesty. His guidance has been instrumental in shaping my understanding of the subject and in developing a disciplined academic mindset.

This work is a humble expression of gratitude to all those who believed in me, supported me, and contributeddirectly or indirectlyto my educational journey. May this achievement reflect a small part of their sacrifices, efforts, and noble intentions.

Abdeljalil Mostefaoui

1 The Set of Real Numbers $\mathbb{R}$

1.1 Summary of Real Numbers

1.1.1 Field Axioms

Definition

The set of real numbers $\mathbb{R}$ is a **field**, meaning it satisfies the following axioms for all $a, b, c \in \mathbb{R}$:

1. **Commutativity of addition:** $a + b = b + a$
2. **Associativity of addition:** $(a+b)+c = a + (b + c)$
3. **Existence of additive identity:** $\exists 0 \in \mathbb{R}$ such that $a + 0 = a$
4. **Existence of additive inverses:** $\forall a \in \mathbb{R}, \exists(-a) \in \mathbb{R}$ such that $a + (-a) = 0$
5. **Commutativity of multiplication:** $ab = ba$
6. **Associativity of multiplication:** $(ab)c = a(bc)$
7. **Existence of multiplicative identity:** $\exists 1 \in \mathbb{R}, 1 \neq 0$ such that $a \cdot 1 = a$
8. **Existence of multiplicative inverses:** $\forall a \in \mathbb{R}, a \neq 0, \exists a^{-1} \in \mathbb{R}$ such that $a \cdot a^{-1} = 1$
9. **Distributivity:** $a(b + c) = ab + ac$

Theorem

Uniqueness of Identities and Inverses:

1. The additive identity 0 is unique.
2. The multiplicative identity 1 is unique.
3. For each $a \in \mathbb{R}$, the additive inverse $-a$ is unique.
4. For each $a \in \mathbb{R}, a \neq 0$, the multiplicative inverse a^{-1} is unique.

Proof. 1. Suppose 0 and $0'$ are both additive identities. Then:

$$0 = 0 + 0' = 0'$$

2. Suppose 1 and $1'$ are both multiplicative identities. Then:

$$1 = 1 \cdot 1' = 1'$$

3. Suppose b and c are both additive inverses of a. Then:

$$b = b + 0 = b + (a + c) = (b + a) + c = 0 + c = c$$

4. Similar to (3).

□

1.1.2 Basic Algebraic Properties

Proposition 1.1. *For all $a, b, c \in \mathbb{R}$:*

1. $a \cdot 0 = 0$
2. $(-a) = (-1) \cdot a$
3. $(-a)(-b) = ab$
4. *If $ab = 0$, then $a = 0$ or $b = 0$ (no zero divisors)*
5. $-(a + b) = (-a) + (-b)$

Proof. 1. $a \cdot 0 = a \cdot (0+0) = a \cdot 0 + a \cdot 0$. Add $-(a \cdot 0)$ to both sides to get $0 = a \cdot 0$.

2. $a + (-1)a = 1 \cdot a + (-1)a = (1 + (-1))a = 0 \cdot a = 0$, so $(-1)a$ is the additive inverse of a.

3. $(-a)(-b) = ((-1)a)((-1)b) = ((-1)(-1))(ab) = 1 \cdot ab = ab$.

□

1.1.3 Order Structure

Definition

The real numbers are equipped with an **order relation** $<$ (less than) satisfying:

1. **Trichotomy:** For any $a, b \in \mathbb{R}$, exactly one of the following holds:
$$a < b, \quad a = b, \quad b < a$$
2. **Transitivity:** If $a < b$ and $b < c$, then $a < c$
3. **Compatibility with addition:** If $a < b$, then $a + c < b + c$ for all $c \in \mathbb{R}$
4. **Compatibility with multiplication:** If $a < b$ and $0 < c$, then $ac < bc$

Definition

We define additional relations:

- $a \leq b$ means $a < b$ or $a = b$
- $a > b$ means $b < a$
- $a \geq b$ means $b \leq a$

Proposition 1.2. ***Basic Order Properties:***

1. *If $a < b$ and $c < 0$, then $ac > bc$*
2. *If $0 < a < b$, then $0 < \frac{1}{b} < \frac{1}{a}$*
3. *If $a < b$ and $c < d$, then $a + c < b + d$*
4. *If $0 \leq a < b$ and $0 \leq c < d$, then $ac < bd$*
5. *For any $a \in \mathbb{R}$, $a^2 \geq 0$, with equality iff $a = 0$*

1.1.4 Absolute Value

Definition

The **absolute value** of a real number x is defined by:
$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Theorem

Properties of Absolute Value: For all $x, y \in \mathbb{R}$:

1. $|x| \geq 0$, and $|x| = 0$ iff $x = 0$
2. $|xy| = |x||y|$
3. $|x + y| \leq |x| + |y|$ (Triangle Inequality)
4. $\big||x| - |y|\big| \leq |x - y|$

5. $|x| < a$ iff $-a < x < a$ for any $a > 0$
6. $|x| > a$ iff $x > a$ or $x < -a$ for any $a > 0$

Proof of Triangle Inequality. Consider four cases:

1. If $x \geq 0$ and $y \geq 0$: $|x + y| = x + y = |x| + |y|$
2. If $x \geq 0$ and $y < 0$:
   - If $x + y \geq 0$, then $|x + y| = x + y \leq x = |x| \leq |x| + |y|$
   - If $x + y < 0$, then $|x + y| = -(x + y) = -x - y \leq -y = |y| \leq |x| + |y|$
3. If $x < 0$ and $y \geq 0$: Symmetric to case 2
4. If $x < 0$ and $y < 0$: $|x + y| = -(x + y) = (-x) + (-y) = |x| + |y|$

□

1.1.5 Intervals

Definition

An **interval** in $\mathbb{R}$ is a set $I \subseteq \mathbb{R}$ such that for all $x, y \in I$ with $x < y$, the entire segment $[x, y]$ is contained in I.

Example

Types of Intervals:

1. **Open interval:** $(a, b) = \{x \in \mathbb{R} : a < x < b\}$
2. **Closed interval:** $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$
3. **Half-open intervals:** $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$, $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$
4. **Infinite intervals:**

$$\begin{aligned}(a, \infty) &= \{x \in \mathbb{R} : x > a\} \\ [a, \infty) &= \{x \in \mathbb{R} : x \geq a\} \\ (-\infty, b) &= \{x \in \mathbb{R} : x < b\} \\ (-\infty, b] &= \{x \in \mathbb{R} : x \leq b\}\end{aligned}$$

5. $\mathbb{R} = (-\infty, \infty)$

Proposition 1.3. ***Properties of Intervals:***

1. *The intersection of two intervals is either empty or an interval*
2. *The union of two intervals may not be an interval*
3. *Every interval is one of the types listed above*

1.1.6 Density of $\mathbb{Q}$ and $\mathbb{R}$

Definition

A set $D \subseteq \mathbb{R}$ is **dense** in $\mathbb{R}$ if between any two distinct real numbers, there exists an element of D.

Theorem

Density of Rational Numbers: The set $\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof. Let $a, b \in \mathbb{R}$ with $a < b$. We need to find $r \in \mathbb{Q}$ such that $a < r < b$.

Since $b - a > 0$, by the Archimedean property, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < b - a$.

Consider the set $S = \{k \in \mathbb{Z} : k > na\}$. By the Archimedean property, S is non-empty. Let $m = \min S$. Then:

$$m > na$$
$$m - 1 \leq na$$

Define $r = \frac{m}{n}$. Then:

$$r = \frac{m}{n} > \frac{na}{n} = a$$

and

$$r = \frac{m}{n} = \frac{m-1}{n} + \frac{1}{n} \leq a + \frac{1}{n} < a + (b - a) = b$$

Thus $a < r < b$. □

Theorem

Density of Irrational Numbers: The set $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.

Proof. Let $a, b \in \mathbb{R}$ with $a < b$. By density of $\mathbb{Q}$, there exists $r \in \mathbb{Q}$ such that $a - \sqrt{2} < r < b - \sqrt{2}$. Then $r + \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$ and $a < r + \sqrt{2} < b$. □

1.1.7 Archimedean Property

Theorem

Archimedean Property: For any real number x, there exists a natural number n such that $n > x$.

Proof. Suppose for contradiction that there exists $x \in \mathbb{R}$ such that $n \leq x$ for all $n \in \mathbb{N}$. Then $\mathbb{N}$ is bounded above by x. By the completeness axiom (to be introduced in Part II), $\mathbb{N}$ has a supremum $M = \sup \mathbb{N}$.

Since $M - 1 < M$, $M - 1$ is not an upper bound of $\mathbb{N}$. Thus there exists $n \in \mathbb{N}$ with $n > M - 1$. But then $n + 1 > M$, contradicting that M is an upper bound. □

Corollary 1.4. ***Consequences of Archimedean Property:***

1. *For any $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$*
2. *For any $x > 0$, there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < x$*
3. *The set $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ has infimum 0*

Example

Example Applications:

1. Show that for any $x > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{2^n} < x$.

   Solution: By Archimedean property, choose n such that $n > -\frac{\ln x}{\ln 2}$.

2. Prove that $\inf\left\{\frac{1}{n} + \frac{1}{m} : n, m \in \mathbb{N}\right\} = 0$.

   Solution: Clearly 0 is a lower bound. For any $\varepsilon > 0$, choose n such that $\frac{1}{n} < \frac{\varepsilon}{2}$. Then $\frac{1}{n} + \frac{1}{n} < \varepsilon$.

1.1.8 Upper and Lower Bounds

Definition

Let $S \subseteq \mathbb{R}$ be a non-empty set.

1. $M \in \mathbb{R}$ is an **upper bound** of S if $x \leq M$ for all $x \in S$
2. $m \in \mathbb{R}$ is a **lower bound** of S if $m \leq x$ for all $x \in S$
3. S is **bounded above** if it has an upper bound
4. S is **bounded below** if it has a lower bound
5. S is **bounded** if it is both bounded above and below

Example

Examples:

1. $S = [0, 1]$: Upper bounds include $1, 2, 3, \ldots$; lower bounds include $0, -1, -2, \ldots$
2. $S = \mathbb{N}$: Bounded below by 1 but not bounded above
3. $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$: Bounded below by 0 and above by 1
4. $S = \emptyset$: Conventionally, every real number is both an upper and lower bound

1.1.9 Supremum and Infimum

Definition

Let $S \subseteq \mathbb{R}$ be non-empty.

1. The **supremum** (least upper bound) of S, denoted $\sup S$, is the smallest upper bound of S
2. The **infimum** (greatest lower bound) of S, denoted $\inf S$, is the largest lower bound of S

Theorem

Characterization of Supremum: For a non-empty set $S \subseteq \mathbb{R}$ and $M \in \mathbb{R}$, the following are equivalent:

1. $M = \sup S$
2. (i) M is an upper bound of S, and (ii) for every $\varepsilon > 0$, there exists $x \in S$ such that $M - \varepsilon < x \leq M$

Proof. $(1 \Rightarrow 2)$: If $M = \sup S$, then M is an upper bound. If (ii) were false, there would exist $\varepsilon > 0$ such that for all $x \in S$, $x \leq M - \varepsilon$. Then $M - \varepsilon$ would be a smaller upper bound, contradicting that M is the least upper bound.

$(2 \Rightarrow 1)$: Condition (i) says M is an upper bound. If $M' < M$ were an upper bound, take $\varepsilon = M - M' > 0$. By (ii), there exists $x \in S$ with $x > M - \varepsilon = M'$, contradicting that M' is an upper bound. □

Theorem

Characterization of Infimum: For a non-empty set $S \subseteq \mathbb{R}$ and $m \in \mathbb{R}$, the following are equivalent:

1. $m = \inf S$
2. (i) m is a lower bound of S, and (ii) for every $\varepsilon > 0$, there exists $x \in S$ such that $m \leq x < m + \varepsilon$

1.1.10 Maximum and Minimum

Definition

Let $S \subseteq \mathbb{R}$ be non-empty.

1. $M \in S$ is the **maximum** of S if $M \geq x$ for all $x \in S$
2. $m \in S$ is the **minimum** of S if $m \leq x$ for all $x \in S$

Remark 1.5. *1. If* $\max S$ *exists, then* $\max S = \sup S$

2. If $\min S$ *exists, then* $\min S = \inf S$

3. The supremum/infimum may exist even when maximum/minimum don't exist

Example

Examples:

1. $S = [0, 1]$: $\sup S = \max S = 1$, $\inf S = \min S = 0$
2. $S = (0, 1)$: $\sup S = 1$, $\inf S = 0$, but neither max nor min exist
3. $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$: $\sup S = \max S = 1$, $\inf S = 0$ but $\min S$ doesn't exist
4. $S = \mathbb{N}$: $\inf S = \min S = 1$, but $\sup S$ doesn't exist (not bounded above)

1.1.11 Completeness Axiom

Theorem

Completeness Axiom (Least Upper Bound Property): Every non-empty subset of $\mathbb{R}$ that is bounded above has a supremum in $\mathbb{R}$.

Corollary 1.6. *Every non-empty subset of* $\mathbb{R}$ *that is bounded below has an infimum in* $\mathbb{R}$.

Proof. Let S be non-empty and bounded below. Consider the set $-S = \{-x : x \in S\}$. Then $-S$ is bounded above, so by completeness, $\sup(-S)$ exists. Then $\inf S = -\sup(-S)$. □

1.1.12 Proof Techniques with ε-Arguments

Example

Example 1: Show that $\sup\{x \in \mathbb{R} : x^2 < 2\} = \sqrt{2}$.
Solution: Let $S = \{x \in \mathbb{R} : x^2 < 2\}$. Clearly $\sqrt{2}$ is an upper bound. Need to show it's the least upper bound.
Given $\varepsilon > 0$, we need $x \in S$ with $\sqrt{2} - \varepsilon < x$. If $\varepsilon \geq \sqrt{2}$, take $x = 0$. Otherwise, consider $x = \sqrt{2} - \frac{\varepsilon}{2}$. Check:

$$x^2 = \left(\sqrt{2} - \frac{\varepsilon}{2}\right)^2 = 2 - \varepsilon\sqrt{2} + \frac{\varepsilon^2}{4} < 2 \quad \text{for small enough } \varepsilon$$

Thus $x \in S$ and $x > \sqrt{2} - \varepsilon$.

Example

Example 2: Find $\inf\left\{\frac{1}{n} + \frac{1}{m} : n, m \in \mathbb{N}\right\}$.
Solution: Let $S = \left\{\frac{1}{n} + \frac{1}{m} : n, m \in \mathbb{N}\right\}$. Clearly 0 is a lower bound. Show $\inf S = 0$.
Given $\varepsilon > 0$, choose $n > \frac{2}{\varepsilon}$. Then $\frac{1}{n} + \frac{1}{n} = \frac{2}{n} < \varepsilon$. Thus there exists $s \in S$ with $s < \varepsilon$, so 0 is the infimum.

1.1.13 Nested Intervals Property

Theorem

Nested Intervals Theorem: Let $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ be a sequence of non-empty closed intervals $I_n = [a_n, b_n]$. Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. The sequence (a_n) is increasing and bounded above by b_1, so it converges to $a = \sup\{a_n\}$. Similarly, (b_n) converges to $b = \inf\{b_n\}$. Since $a_n \leq b_n$ for all n, we have $a \leq b$. For any n, $a_n \leq a \leq b \leq b_n$, so $[a, b] \subseteq I_n$. Thus $[a, b] \subseteq \bigcap_{n=1}^{\infty} I_n$. □

Remark 1.7. *The intersection may be a single point, as with* $I_n = \left[0, \frac{1}{n}\right]$.

1.2 List of Exercises

Below are the exercises as presented in the source document. Click the button to jump to the solution.

Exercise

Find the supremum and infimum, if they exist, of the set E in S:

1. $E = \{(-1)^n + \frac{1}{n} \mid n \in \mathbb{N}^*\}$, $S = \mathbb{Q}$.
2. $E = \{\cos \frac{2n\pi}{7} \mid n \in \mathbb{Z}\}$, $S = \mathbb{R}$.
3. $E = \{\frac{1}{2x-1} \mid x \in [-1, 2[\backslash\{\frac{1}{2}\}\}$, $S = \mathbb{R}$.
4. $E = \{x^2 + y^2 \mid (x, y) \in \mathbb{R}^2, xy = 1\}$, $S = \mathbb{R}$.

Go to Proof I

Exercise

Let $n \in \mathbb{N}^*$ such that n is not the square of any natural number. Prove that:

1. $\sqrt{n} \notin \mathbb{Q}$.
2. $\sqrt{2} + \sqrt{3} \notin \mathbb{Q}$.

Go to Proof II

Exercise

Prove that:

1. $\forall n \in \mathbb{N}^*, \forall x \in \mathbb{R} : [\frac{[nx]}{n}] = [x]$.
2. $\forall (x, y) \in \mathbb{R}^2 : [x] + [y] \leq [x + y] \leq [x] + [y] + 1$.

(Where $[x]$ represents the integer part of x). Go to Proof III

Exercise

Prove that:

1. $\forall n \in \mathbb{N} : \sum_{k=0}^{n} C_n^k = 2^n$.
2. $\forall n \in \mathbb{N}^* : \sum_{k=0}^{n} (-1)^k C_n^k = 0$.
3. Property of Archimedes: $\forall \epsilon > 0, \forall a \in \mathbb{R}, \exists n \in \mathbb{N}^* : a < n\epsilon$.

Go to Proof IV

Exercise

Prove that 0 is an accumulation point for the set $E = \{\frac{1}{n} \mid n \in \mathbb{N}^*\}$. Go to Proof V

Exercise

Let E be a non-empty bounded above subset of $\mathbb{R}$ and $a = \sup E$.

1. Prove that a is an adherent point of E.
2. Prove that if $a \notin E$, then a is an accumulation point of E.

Go to Proof VI

1.3 Solutions of Exercises

Solution to Exercise I

Back to Exercise I

Rigorous Solution

Part 1: $E_1 = \{(-1)^n + \frac{1}{n} \mid n \in \mathbb{N}^*\}$ in $S = \mathbb{Q}$.
Analysis: We split the sequence $u_n = (-1)^n + \frac{1}{n}$ into even and odd terms.

- If $n = 2k$ (even): $u_{2k} = 1 + \frac{1}{2k}$. This is a decreasing subsequence converging to 1.
- If $n = 2k+1$ (odd): $u_{2k+1} = -1 + \frac{1}{2k+1}$. This is a decreasing subsequence converging to -1.

Calculating values:

- $n = 1$: $-1 + 1 = 0$
- $n = 2$: $1 + 1/2 = 3/2$ (Max of even terms)
- $n = 3$: $-1 + 1/3 = -2/3$
- $n = 4$: $1 + 1/4 = 5/4$

Supremum: For even n, $u_n = 1 + \frac{1}{n}$. Since $n \geq 2$ for even n, the maximum value is at $n = 2$, which is $3/2$. For odd n, $u_n = -1 + \frac{1}{n}$. The maximum is at $n = 1$, which is 0. Since $3/2 > 0$, $\sup(E_1) = \max(E_1) = \frac{3}{2}$. Since $\frac{3}{2} \in \mathbb{Q}$, it exists in S.
Infimum: For even n, $u_{2k} \to 1$. For odd n, $u_{2k+1} = -1 + \frac{1}{2k+1}$. Since $n \geq 1$, $\frac{1}{n} > 0$, so $u_n > -1$. However, as $n \to \infty$, u_n approaches -1. Thus, the lower bound is -1. Since $-1 \in \mathbb{Q}$, $\inf(E_1) = -1$. (Note: $-1 \notin E_1$, so it is not a minimum).
Part 2: $E_2 = \{\cos(\frac{2n\pi}{7}) \mid n \in \mathbb{Z}\}$ in $S = \mathbb{R}$.
Analysis: The cosine function is periodic with 2π. The argument is $\frac{2n\pi}{7}$. The values repeat when $\frac{2n\pi}{7}$ changes by $2\pi k$, i.e., $2n/7 = 2k \implies n = 7k$. Thus, we only need to evaluate $n \in \{0, 1, 2, 3, 4, 5, 6\}$. Recall $\cos(x) = \cos(2\pi - x)$.

- $n = 0 : \cos(0) = 1$.
- $n = 1, 6 : \cos(2\pi/7) = \cos(12\pi/7)$.
- $n = 2, 5 : \cos(4\pi/7) = \cos(10\pi/7)$.
- $n = 3, 4 : \cos(6\pi/7) = \cos(8\pi/7)$.

The set is finite: $E_2 = \{1, \cos(\frac{2\pi}{7}), \cos(\frac{4\pi}{7}), \cos(\frac{6\pi}{7})\}$. Cosine is decreasing on $[0, \pi]$. Since $0 < \frac{2\pi}{7} < \frac{4\pi}{7} < \frac{6\pi}{7} < \pi$, we have: $1 > \cos(\frac{2\pi}{7}) > \cos(\frac{4\pi}{7}) > \cos(\frac{6\pi}{7})$.
Conclusion: $\sup(E_2) = \max(E_2) = 1$. $\inf(E_2) = \min(E_2) = \cos(\frac{6\pi}{7})$.
Part 3: $E_3 = \{\frac{1}{2x-1} \mid x \in [-1, 2) \setminus \{1/2\}\}$.
Let $f(x) = \frac{1}{2x-1}$. We analyze the intervals separately.

- **Interval A:** $x \in [-1, 1/2)$. $2x - 1$ ranges from $2(-1) - 1 = -3$ to $2(1/2) - 1 \to 0^-$. Reciprocal: $\frac{1}{-3} \geq f(x) > -\infty$. Range A: $(-\infty, -1/3]$.
- **Interval B:** $x \in (1/2, 2)$. $2x - 1$ ranges from 0^+ to $2(2) - 1 = 3$. Reciprocal: $+\infty > f(x) > 1/3$. Range B: $(1/3, +\infty)$.

$E_3 = (-\infty, -1/3] \cup (1/3, +\infty)$.
Conclusion: $\sup(E_3)$ does not exist (unbounded above). $\inf(E_3)$ does not exist (unbounded below).
Part 4: $E_4 = \{x^2 + y^2 \mid xy = 1\}$. Substitute $y = 1/x$. Since $x, y \in \mathbb{R}$ and $xy = 1$, $x \neq 0$. $E_4 = \{x^2 + \frac{1}{x^2} \mid x \in \mathbb{R}^*\}$. By AM-GM inequality (or simple algebra $(x^2 - 1)^2 \geq 0$): $x^2 + \frac{1}{x^2} \geq 2\sqrt{x^2 \cdot \frac{1}{x^2}} = 2$. Equality holds when $x^2 = 1/x^2 \implies x^4 = 1 \implies x = \pm 1$. The function grows unbounded as $x \to \infty$ or $x \to 0$.
Conclusion: $\sup(E_4) = +\infty$ (Does not exist in $\mathbb{R}$). $\inf(E_4) = 2$ (and it is a minimum).

Solution to Exercise II

Back to Exercise II

Rigorous Solution

1. Prove $\sqrt{n} \notin \mathbb{Q}$ if n is not a perfect square.
Proof by Contradiction: Assume $\sqrt{n} \in \mathbb{Q}$. Then $\sqrt{n} = \frac{p}{q}$ where $p, q \in \mathbb{N}^*$, and $\gcd(p, q) = 1$ (irreducible fraction).

$$n = \frac{p^2}{q^2} \implies p^2 = nq^2$$

Consider the prime factorization of both sides. In p^2, every prime factor appears with an even exponent. In q^2, every prime factor appears with an even exponent. For nq^2 to equal p^2, the prime factorization of n must also have all even exponents (so that total exponents remain even). If all prime factors of n have even exponents, then n is a perfect square ($n = k^2$). This contradicts the hypothesis that n is not a square. Therefore, $\sqrt{n} \notin \mathbb{Q}$. □

2. Prove $\sqrt{2} + \sqrt{3} \notin \mathbb{Q}$.
Proof by Contradiction: Assume $x = \sqrt{2} + \sqrt{3} \in \mathbb{Q}$. Since $\mathbb{Q}$ is a field, $x^2 \in \mathbb{Q}$.

$$x^2 = (\sqrt{2} + \sqrt{3})^2 = 2 + 2\sqrt{6} + 3 = 5 + 2\sqrt{6}$$

We can isolate $\sqrt{6}$:

$$2\sqrt{6} = x^2 - 5 \implies \sqrt{6} = \frac{x^2 - 5}{2}$$

Since $x \in \mathbb{Q}$, then $x^2 \in \mathbb{Q}$, and $\frac{x^2-5}{2} \in \mathbb{Q}$. This implies $\sqrt{6} \in \mathbb{Q}$. However, 6 is not a perfect square ($2^2 = 4 < 6 < 3^2 = 9$). By Part 1, $\sqrt{6} \notin \mathbb{Q}$. This is a contradiction. Therefore, $\sqrt{2} + \sqrt{3} \notin \mathbb{Q}$. □

Solution to Exercise III

Back to Exercise III

Rigorous Solution

1. Prove $[\frac{[nx]}{n}] = [x]$.
Let $k = [x]$. By definition of the integer part:

$$k \leq x < k + 1$$

Multiplying by n (since $n > 0$):

$$nk \leq nx < nk + n$$

The integer part $[nx]$ must satisfy:

$$nk \leq [nx] \leq nk + n - 1$$

(Since $[nx]$ is an integer and strictly less than $nk + n$). Now, divide the inequality by n:

$$k \leq \frac{[nx]}{n} \leq k + \frac{n-1}{n}$$

Since $\frac{n-1}{n} = 1 - \frac{1}{n} < 1$, we have:

$$k \leq \frac{[nx]}{n} < k + 1$$

By definition, the integer part of a number y such that $k \leq y < k + 1$ is k. Therefore:

$$\left[\frac{[nx]}{n}\right] = k = [x]$$

□

2. Prove $[x] + [y] \leq [x + y] \leq [x] + [y] + 1$.
Let $[x] = n$ and $[y] = m$. Let $x = n + \alpha$ where $0 \leq \alpha < 1$. Let $y = m + \beta$ where $0 \leq \beta < 1$. Then $x + y = n + m + (\alpha + \beta)$.
We distinguish two cases for $\alpha + \beta$:
Case 1: $0 \leq \alpha + \beta < 1$. Then $[x + y] = [n + m + (\alpha + \beta)] = n + m$. So, $[x + y] = [x] + [y]$. The inequality becomes: $n + m \leq n + m \leq n + m + 1$, which is true.
Case 2: $1 \leq \alpha + \beta < 2$ (since $\alpha, \beta < 1$). Then $[x + y] = [n + m + (\alpha + \beta)] = n + m + 1$. The inequality becomes: $n + m \leq n + m + 1 \leq n + m + 1$, which is true.
In both cases, the inequality holds. □

Solution to Exercise IV

Back to Exercise IV

Rigorous Solution

1. Prove $\sum_{k=0}^{n} C_n^k = 2^n$.
We use the Binomial Theorem:

$$(a+b)^n = \sum_{k=0}^{n} C_n^k a^k b^{n-k}$$

Let $a = 1$ and $b = 1$.

$$(1+1)^n = \sum_{k=0}^{n} C_n^k (1)^k (1)^{n-k}$$

$$2^n = \sum_{k=0}^{n} C_n^k$$

□

2. Prove $\sum_{k=0}^{n} (-1)^k C_n^k = 0$.
Using the Binomial Theorem again, let $a = -1$ and $b = 1$.

$$(-1+1)^n = \sum_{k=0}^{n} C_n^k (-1)^k (1)^{n-k}$$

$$0^n = \sum_{k=0}^{n} (-1)^k C_n^k$$

For $n \in \mathbb{N}^*$, $0^n = 0$. Thus the sum is 0. □

3. Property of Archimedes: $\forall \epsilon > 0, \forall a \in \mathbb{R}, \exists n \in \mathbb{N}^* : a < n\epsilon$.
Proof by Contradiction: Assume the negation: $\exists \epsilon > 0, \exists a \in \mathbb{R}$ such that $\forall n \in \mathbb{N}^*, n\epsilon \leq a$. This implies that the set $A = \{n\epsilon \mid n \in \mathbb{N}^*\}$ is bounded above by a. By the Least Upper Bound Axiom (Completeness of $\mathbb{R}$), A has a supremum. Let $S = \sup(A)$. Since S is the least upper bound, $S - \epsilon$ is not an upper bound (because $S - \epsilon < S$). Therefore, there exists an element in A, say $n_0\epsilon$, such that:

$$n_0\epsilon > S - \epsilon$$

Rearranging gives:

$$(n_0 + 1)\epsilon > S$$

However, $(n_0 + 1)\epsilon$ is also in A (since $n_0 + 1 \in \mathbb{N}^*$). This states that there is an element in A greater than the supremum S, which is a contradiction. Therefore, the assumption is false, and the Archimedean property holds. □

Solution to Exercise V

Back to Exercise V

Rigorous Solution

Prove that 0 is an accumulation point for $E = \{\frac{1}{n} \mid n \in \mathbb{N}^*\}$.

Definition

Accumulation Point A point x is an accumulation point of a set E if every neighborhood V of x contains at least one point of E distinct from x. Formally: $\forall \epsilon > 0, (V_\epsilon(x) \cap E) \setminus \{x\} \neq \emptyset$.

Proof: Let $\epsilon > 0$ be arbitrary. Consider the neighborhood $V = (-\epsilon, \epsilon)$. We need to show there exists an element $e \in E$ such that $e \in V$ and $e \neq 0$. Since $e = 1/n$, $e \neq 0$ is always true. We just need $1/n < \epsilon$. By the **Archimedean Property** (proved in IV.3), for any $\epsilon > 0$ and $a = 1$, there exists $n \in \mathbb{N}^*$ such that:

$$1 < n\epsilon \implies \frac{1}{n} < \epsilon$$

Since $n \in \mathbb{N}^*$, $\frac{1}{n} > 0$. Thus, $0 < \frac{1}{n} < \epsilon$. This implies $\frac{1}{n} \in (-\epsilon, \epsilon)$. So, the intersection is non-empty for any ϵ. Therefore, 0 is an accumulation point of E. □

Solution to Exercise VI

Back to Exercise VI

Rigorous Solution

et $E \subset \mathbb{R}$ be bounded above and $a = \sup E$.

1. Prove that a is an adherent point of E.

Definition

Adherent Point x is an adherent point of E if every neighborhood of x intersects E. $\forall \epsilon > 0, V_\epsilon(x) \cap E \neq \emptyset$.

Proof: Let $\epsilon > 0$. Since $a = \sup E$, a is the least upper bound. This implies $a - \epsilon$ is *not* an upper bound of E. Therefore, there must exist some $x \in E$ such that $x > a - \epsilon$. Since a is an upper bound, $x \leq a$. So, $a - \epsilon < x \leq a < a + \epsilon$. Thus, $x \in (a - \epsilon, a + \epsilon)$, which means $V_\epsilon(a) \cap E \neq \emptyset$. Therefore, a is adherent to E. □

2. Prove that if $a \notin E$, then a is an accumulation point of E.

Proof: By definition of an accumulation point, we must show that for any $\epsilon > 0$, the intersection $(a - \epsilon, a + \epsilon) \cap E$ contains a point distinct from a. From Part 1, we know the intersection is not empty. Let $x \in (a - \epsilon, a + \epsilon) \cap E$. Since the hypothesis states $a \notin E$, and $x \in E$, it implies $x \neq a$. Therefore, every neighborhood of a contains a point $x \in E$ with $x \neq a$. Thus, a is an accumulation point. □

2 Complex Numbers

2.1 Summary of Complex Numbers

2.1.1 Algebraic Form

Definition

A **complex number** is an expression of the form

$$z = a + bi$$

where $a, b \in \mathbb{R}$ and i is the imaginary unit satisfying $i^2 = -1$.

- a is called the **real part**, denoted $\Re(z) = a$
- b is called the **imaginary part**, denoted $\Im(z) = b$
- The set of all complex numbers is denoted $\mathbb{C}$

Proposition 2.1. ***Operations in Algebraic Form:*** *For $z = a + bi, w = c + di \in \mathbb{C}$:*

1. ***Addition:*** $z + w = (a + c) + (b + d)i$
2. ***Multiplication:*** $zw = (ac - bd) + (ad + bc)i$
3. ***Conjugate:*** $\overline{z} = a - bi$
4. ***Modulus:*** $|z| = \sqrt{a^2 + b^2}$

Proposition 2.2. ***Properties of Conjugate and Modulus:*** *For $z, w \in \mathbb{C}$:*

1. $\overline{\overline{z}} = z$
2. $\overline{z + w} = \overline{z} + \overline{w}$
3. $\overline{zw} = \overline{z} \cdot \overline{w}$
4. $|z|^2 = z\overline{z}$
5. $|zw| = |z||w|$
6. $|z + w| \leq |z| + |w|$ *(Triangle Inequality)*

2.1.2 Trigonometric Form

Definition

Let $z = a + bi \in \mathbb{C} \setminus \{0\}$. Define:

- **Modulus:** $r = |z| = \sqrt{a^2 + b^2}$
- **Argument:** $\theta = \arg(z)$ such that

$$a = r\cos\theta, \quad b = r\sin\theta$$

  The argument is determined modulo 2π.

Then the **trigonometric form** of z is:

$$z = r(\cos\theta + i\sin\theta)$$

Proposition 2.3. ***Operations in Trigonometric Form:*** *Let $z = r(\cos\theta + i\sin\theta)$, $w = s(\cos\phi + i\sin\phi)$.*

1. ***Multiplication:*** $zw = rs[\cos(\theta + \phi) + i\sin(\theta + \phi)]$
2. ***Division:*** $\frac{z}{w} = \frac{r}{s}[\cos(\theta - \phi) + i\sin(\theta - \phi)]$, $w \neq 0$
3. ***Reciprocal:*** $z^{-1} = \frac{1}{r}[\cos(-\theta) + i\sin(-\theta)]$

2.1.3 Exponential Form

Definition

For $\theta \in \mathbb{R}$, define:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

This is **Euler's Formula**. Then any $z \in \mathbb{C} \setminus \{0\}$ can be written as:

$$z = re^{i\theta}$$

where $r = |z|$ and $\theta = \arg(z)$.

Proposition 2.4. ***Properties of Exponential Form:***

1. $e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}$
2. $\frac{e^{i\theta}}{e^{i\phi}} = e^{i(\theta-\phi)}$
3. $(e^{i\theta})^n = e^{in\theta}$ *for* $n \in \mathbb{Z}$
4. $\overline{e^{i\theta}} = e^{-i\theta}$
5. $|e^{i\theta}| = 1$

2.1.4 De Moivre's Theorem and Euler's Formula

Theorem

De Moivre's Theorem: For any $\theta \in \mathbb{R}$ and $n \in \mathbb{Z}$:

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

In exponential form: $(e^{i\theta})^n = e^{in\theta}$.

Proof. Using induction for $n \geq 0$. For negative n, use the reciprocal formula. □

Theorem

Euler's Formulas:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

2.1.5 Linearization of Trigonometric Polynomials

Definition

Linearization is the process of expressing powers of trigonometric functions as sums of trigonometric functions of multiple angles.

Example

Examples:

1. $\cos^2\theta = \frac{1+\cos 2\theta}{2}$
2. $\sin^2\theta = \frac{1-\cos 2\theta}{2}$
3. $\cos^3\theta = \frac{3\cos\theta+\cos 3\theta}{4}$
4. $\sin^3\theta = \frac{3\sin\theta-\sin 3\theta}{4}$

Proposition 2.5. *To linearize* $\cos^n\theta$ *or* $\sin^n\theta$*:*

1. *Write* $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$, $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$
2. *Expand using binomial theorem*
3. *Group terms to obtain linear combination of* $\cos(k\theta)$ *or* $\sin(k\theta)$

2.1.6 Square Roots of Complex Numbers

Theorem

Let $z = a + bi \in \mathbb{C}$. The square roots of z are given by:

$$w = \pm\left(\sqrt{\frac{|z|+a}{2}} + i \cdot \frac{b}{|b|}\sqrt{\frac{|z|-a}{2}}\right), \quad b \neq 0$$

For $b = 0$, the formula reduces to real square roots.

Proof. Let $w = x + iy$ such that $w^2 = z$. Then:

$$x^2 - y^2 = a, \quad 2xy = b$$

Solving this system yields the formula. □

Example

Example: Find square roots of $3 + 4i$.
Solution: $|3 + 4i| = 5$. Using formula:

$$w = \pm\left(\sqrt{\frac{5+3}{2}} + i \cdot \frac{4}{|4|}\sqrt{\frac{5-3}{2}}\right) = \pm(2 + i)$$

Verify: $(2 + i)^2 = 4 + 4i + i^2 = 3 + 4i$.

2.1.7 Roots of Unity

Definition

The **n-th roots of unity** are solutions to $z^n = 1$. They are:

$$\zeta_n^k = e^{2\pi ik/n}, \quad k = 0, 1, \ldots, n-1$$

- $\zeta_n = e^{2\pi i/n}$ is the **primitive n-th root of unity**
- The roots are equally spaced on the unit circle

Proposition 2.6. ***Properties of Roots of Unity:***

1. $\zeta_n^0 = 1$
2. $\zeta_n^k \cdot \zeta_n^m = \zeta_n^{k+m}$ *(indices modulo n)*
3. $\overline{\zeta_n^k} = \zeta_n^{-k} = \zeta_n^{n-k}$
4. $\sum_{k=0}^{n-1} \zeta_n^k = 0$ *for* $n > 1$

2.1.8 Complex Exponential and Logarithm

Definition

For $z = x + iy \in \mathbb{C}$, the **complex exponential** is:

$$e^z = e^x(\cos y + i \sin y)$$

Proposition 2.7. ***Properties:***

1. $e^{z+w} = e^z e^w$
2. $|e^z| = e^{\Re(z)}$
3. $\arg(e^z) = \Im(z) \mod 2\pi$
4. $e^z = 1$ *iff* $z = 2\pi i k$ *for some* $k \in \mathbb{Z}$

Definition

The **complex logarithm** is multi-valued:

$$\log z = \ln |z| + i \arg(z)$$

where $\arg(z)$ can be chosen modulo 2π. The **principal value** is:

$$\operatorname{Log} z = \ln |z| + i \operatorname{Arg}(z), \quad \operatorname{Arg}(z) \in (-\pi, \pi]$$

2.1.9 Complex Powers

Definition

For $z, w \in \mathbb{C}$ with $z \neq 0$:

$$z^w = e^{w \log z}$$

This is multi-valued unless $w \in \mathbb{Z}$.

Example

Examples:

1. $i^i = e^{i \log i} = e^{i(\frac{\pi i}{2} + 2\pi i k)} = e^{-\pi/2 - 2\pi k}$
2. $\sqrt{1} = \pm 1$ (two values)

2.1.10 Geometry of Complex Numbers

Proposition 2.8. ***Geometric Interpretations:***

1. *Addition: Vector addition in the plane*
2. *Multiplication: Rotation and scaling*
3. *Conjugate: Reflection across real axis*
4. *Modulus: Distance from origin*

Theorem

Polar Coordinates: Multiplication by $e^{i\theta}$ rotates by angle θ. Multiplication by $r > 0$ scales by factor r.

2.1.11 Fundamental Theorem of Algebra

Theorem

Fundamental Theorem of Algebra: Every non-constant polynomial with complex coefficients has at least one complex root.

Corollary 2.9. *A polynomial of degree n with complex coefficients has exactly n roots in $\mathbb{C}$ (counting multiplicities).*

2.1.12 Complex Conjugate Root Theorem

Theorem

Complex Conjugate Root Theorem: If $P(z)$ is a polynomial with real coefficients and z_0 is a root, then $\overline{z_0}$ is also a root with the same multiplicity.

Proof. For real coefficients, $P(\overline{z}) = \overline{P(z)}$. Thus if $P(z_0) = 0$, then $P(\overline{z_0}) = 0$. □

2.2 List of Exercises

Exercise

Let $(a, b, c) \in \mathbb{C}^3$. Prove that:

1. $|1 + a| + |a + b| + |b + c| + |c| \geq 1$.
2. If $|a| = |b| = 1$ and $a \neq b$, prove that $\forall z \in \mathbb{C} : \frac{z+ab\overline{z}-(a+b)}{a-b} \in i\mathbb{R}$.
3. $|a + b|^2 + |a - b|^2 = 2(|a|^2 + |b|^2)$ (Parallelogram Law).

Go to Proof I

Exercise

Find the points of the complex plane which satisfy the following conditions:

1. $|z - 4| < 1$.
2. $\arg(z - 4) = \frac{\pi}{4}$.
3. $|z - 4| + |z + 4| = 16$.
4. $\frac{5z-2}{z-1}$ is purely imaginary.

Go to Proof II

Exercise

Let $\theta \in \mathbb{R}$.

1. Using Newton's binomial and de Moivre's formulas, write both $\cos 5\theta$ and $\sin 5\theta$ in terms of $\cos\theta$ and $\sin\theta$.
2. Using Newton's binomial and Euler's formulas, write both $\cos^5\theta$ and $\sin^5\theta$ as a linear expression.

Go to Proof III

Exercise

Calculate the following sums:

1. $S_n = \sum_{k=0}^{n} \sin(k\theta)$.
2. $S'_n = \sum_{k=0}^{n} C_n^k \cos(k\theta)$.

Go to Proof IV

Exercise

Solve in $\mathbb{C}$ the following equations:

1. $z^2 = 6 - 8i$.
2. $z^4 + 2 = 0$.
3. $(z + i)^n = (z - i)^n$, $n \geq 1$.

Go to Proof V

Exercise

Let $\alpha \in]-\frac{\pi}{2}, \frac{\pi}{2}[$, $n \in \mathbb{N}^*$, and $\lambda \in \mathbb{C}$ such that $\lambda \neq 1, |\lambda| = 1$.

1. Verify that $\frac{1+i\tan\alpha}{1-i\tan\alpha} = e^{2i\alpha}$.
2. Show that $\frac{i(1-\lambda)}{1+\lambda} = \frac{\mathrm{Im}(\lambda)}{1+\mathrm{Re}(\lambda)}$.
3. Deduce the complex numbers z, solutions of $(\frac{1+iz}{1-iz})^n = \frac{1+i\tan\alpha}{1-i\tan\alpha}$.

Go to Proof VI

Exercise

Let $n \in \mathbb{N}^*$ and $z_0, z_1, ..., z_{n-1}$ be the n-th roots of 1.

1. Calculate the sum $S_q = \sum_{k=0}^{n-1} z_k^q$ for each natural number q.
2. Show that for each $z \in \mathbb{C}$ we have $\sum_{k=0}^{n-1}(z + e^{i\frac{2k\pi}{n}})^n = n(z^n + 1)$.

Go to Proof VII

2.3 Solutions of Exercises

Solution to Exercise I

Back to Exercise I

Rigorous Solution

1. Prove $|1+a|+|a+b|+|b+c|+|c| \geq 1$.
We use the triangle inequality: $|x|+|y| \geq |x+y|$. We can group the terms as follows:

$$(|1+a|+|a+b|)+(|b+c|+|c|)$$

Note that $|x| = |-x|$. Thus $|a+b| = |-a-b|$ and $|c| = |-c|$. Using triangle inequality on the first pair:

$$|1+a|+|-a-b| \geq |(1+a)+(-a-b)| = |1-b|$$

Using triangle inequality on the second pair:

$$|b+c|+|-c| \geq |(b+c)+(-c)| = |b|$$

Now sum the results:

$$\text{LHS} \geq |1-b|+|b|$$

Apply triangle inequality again:

$$|1-b|+|b| \geq |(1-b)+b| = |1| = 1$$

Therefore, the inequality holds. □

2. Prove $W = \frac{z+ab\bar{z}-(a+b)}{a-b} \in i\mathbb{R}$.
A complex number W is purely imaginary if and only if $\overline{W} = -W$. Given: $|a| = |b| = 1 \implies \bar{a} = 1/a, \bar{b} = 1/b$. Calculate $\overline{W}$:

$$\overline{W} = \frac{\bar{z}+\bar{a}\bar{b}z-(\bar{a}+\bar{b})}{\bar{a}-\bar{b}}$$

Substitute $\bar{a} = 1/a, \bar{b} = 1/b$:

$$\overline{W} = \frac{\bar{z}+\frac{1}{ab}z-(\frac{1}{a}+\frac{1}{b})}{\frac{1}{a}-\frac{1}{b}}$$

Multiply numerator and denominator by ab:

$$\overline{W} = \frac{ab\bar{z}+z-(b+a)}{b-a} = \frac{z+ab\bar{z}-(a+b)}{-(a-b)}$$

$$\overline{W} = -\left(\frac{z+ab\bar{z}-(a+b)}{a-b}\right) = -W$$

Since $\overline{W} = -W$, $W \in i\mathbb{R}$. □

3. Parallelogram Law: $|a+b|^2+|a-b|^2 = 2(|a|^2+|b|^2)$.
Recall that $|z|^2 = z\bar{z}$.

$$\begin{aligned}\text{LHS} &= (a+b)(\overline{a+b})+(a-b)(\overline{a-b}) \\ &= (a+b)(\bar{a}+\bar{b})+(a-b)(\bar{a}-\bar{b})\end{aligned}$$

Expand terms:

$$= (a\bar{a}+a\bar{b}+b\bar{a}+b\bar{b})+(a\bar{a}-a\bar{b}-b\bar{a}+b\bar{b})$$

Simplify:

$$\begin{aligned}&= |a|^2+|b|^2+a\bar{b}+b\bar{a}+|a|^2+|b|^2-a\bar{b}-b\bar{a} \\ &= 2|a|^2+2|b|^2 = 2(|a|^2+|b|^2)\end{aligned}$$

□

Solution to Exercise II

Back to Exercise II

Rigorous Solution

1. $|z-4|<1$ This represents the set of points $M(z)$ whose distance from the point $A(4,0)$ is strictly less than 1. **Result:** Open disk centered at $(4,0)$ with radius $R=1$.

2. $\arg(z-4)=\frac{\pi}{4}$ This represents the set of points $M(z)$ such that the vector $\vec{AM}$ makes an angle of $\pi/4$ with the real axis (A is $(4,0)$). **Result:** Half-line starting from $(4,0)$ (excluding the point itself) with slope 1 (angle 45°).

3. $|z-4|+|z+4|=16$ Let $F_1(4,0)$ and $F_2(-4,0)$. The condition is $MF_1+MF_2=16$. Since the distance $F_1F_2=8$ and $16>8$, this is the definition of an ellipse. Major axis $2a=16 \implies a=8$. Focal distance $2c=8 \implies c=4$. Minor axis $b^2=a^2-c^2=64-16=48 \implies b=\sqrt{48}$. **Result:** Ellipse with foci $(\pm 4,0)$. Equation: $\frac{x^2}{64}+\frac{y^2}{48}=1$.

4. $\frac{5z-2}{z-1}\in i\mathbb{R}$ Let $Z=\frac{5z-2}{z-1}$. For Z to be purely imaginary, $\mathrm{Re}(Z)=0$ (and $z\neq 1$). Let $z=x+iy$.

$$Z=\frac{5(x+iy)-2}{(x-1)+iy}=\frac{(5x-2)+i5y}{(x-1)+iy}\cdot\frac{(x-1)-iy}{(x-1)-iy}$$

The real part of the numerator is:

$$(5x-2)(x-1)+(5y)(y)=5x^2-5x-2x+2+5y^2$$

$$\mathrm{Re}(Z)=0 \iff 5x^2+5y^2-7x+2=0$$

Divide by 5:

$$x^2+y^2-\frac{7}{5}x+\frac{2}{5}=0$$

Complete the square for x:

$$(x-\frac{7}{10})^2-\frac{49}{100}+y^2+\frac{40}{100}=0$$

$$(x-0.7)^2+y^2=\frac{9}{100}=(0.3)^2$$

Result: Circle with center $(0.7,0)$ and radius 0.3, excluding point $z=1$ (though $z=1$ is not on this circle anyway).

Solution to Exercise III

Back to Exercise III

Rigorous Solution

1. Expansion of $\cos 5\theta$ and $\sin 5\theta$ By de Moivre's formula: $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$. By Binomial Expansion $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$. Let $c = \cos\theta, s = \sin\theta$.

$$(c+is)^5 = c^5 + 5c^4(is) + 10c^3(is)^2 + 10c^2(is)^3 + 5c(is)^4 + (is)^5$$

$$= c^5 + i5c^4s - 10c^3s^2 - i10c^2s^3 + 5cs^4 + is^5$$

Separate Real and Imaginary parts:

$$\text{Re} : \cos 5\theta = c^5 - 10c^3s^2 + 5cs^4$$

$$\text{Im} : \sin 5\theta = 5c^4s - 10c^2s^3 + s^5$$

2. Linearization of $\cos^5\theta$ Using Euler: $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$.

$$\cos^5\theta = \frac{1}{32}(e^{i\theta} + e^{-i\theta})^5$$

$$= \frac{1}{32}(e^{i5\theta} + 5e^{i3\theta} + 10e^{i\theta} + 10e^{-i\theta} + 5e^{-i3\theta} + e^{-i5\theta})$$

Group conjugates:

$$= \frac{1}{32}[(e^{i5\theta} + e^{-i5\theta}) + 5(e^{i3\theta} + e^{-i3\theta}) + 10(e^{i\theta} + e^{-i\theta})]$$

$$= \frac{1}{32}[2\cos 5\theta + 10\cos 3\theta + 20\cos\theta]$$

$$\cos^5\theta = \frac{1}{16}\cos 5\theta + \frac{5}{16}\cos 3\theta + \frac{5}{8}\cos\theta$$

Similar method applies for $\sin^5\theta$.

Solution to Exercise IV

Back to Exercise IV

Rigorous Solution

1. $S_n = \sum_{k=0}^{n} \sin(k\theta)$ This is the imaginary part of the geometric series $G_n = \sum_{k=0}^{n} (e^{i\theta})^k$. If $\theta = 2m\pi$, $S_n = 0$. Otherwise:

$$G_n = \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}$$

Use half-angle trick: $1 - e^{ix} = -2i \sin(x/2) e^{ix/2}$.

$$G_n = \frac{-2i \sin(\frac{(n+1)\theta}{2}) e^{i\frac{(n+1)\theta}{2}}}{-2i \sin(\frac{\theta}{2}) e^{i\frac{\theta}{2}}} = \frac{\sin(\frac{(n+1)\theta}{2})}{\sin(\frac{\theta}{2})} e^{i\frac{n\theta}{2}}$$

$S_n = \mathrm{Im}(G_n) = \frac{\sin(\frac{(n+1)\theta}{2}) \sin(\frac{n\theta}{2})}{\sin(\frac{\theta}{2})}$.

2. $S'_n = \sum_{k=0}^{n} C_n^k \cos(k\theta)$ Consider the sum $Z = \sum_{k=0}^{n} C_n^k e^{ik\theta} = \sum_{k=0}^{n} C_n^k (e^{i\theta})^k (1)^{n-k}$. By Binomial theorem:

$$Z = (1 + e^{i\theta})^n$$

Factor out $e^{i\theta/2}$:

$$Z = [e^{i\theta/2}(e^{-i\theta/2} + e^{i\theta/2})]^n = [e^{i\theta/2}(2\cos\frac{\theta}{2})]^n$$

$$Z = 2^n \cos^n(\frac{\theta}{2}) e^{in\theta/2}$$

$S'_n = \mathrm{Re}(Z) = 2^n \cos^n(\frac{\theta}{2}) \cos(\frac{n\theta}{2})$.

Solution to Exercise V

Back to Exercise V

Rigorous Solution

1. $z^2 = 6 - 8i$ Let $z = x + iy$. $z^2 = x^2 - y^2 + 2xyi$. System: 1) $x^2 - y^2 = 6$ 2) $2xy = -8 \implies xy = -4$ 3) $x^2 + y^2 = |6 - 8i| = \sqrt{36 + 64} = 10$. Add (1) and (3): $2x^2 = 16 \implies x = \pm\sqrt{8} = \pm 2\sqrt{2}$. From (2), y has opposite sign to x. If $x = 2\sqrt{2}, y = -4/(2\sqrt{2}) = -\sqrt{2}$. Solutions: $z = \pm(2\sqrt{2} - i\sqrt{2})$.

2. $z^4 + 2 = 0 \implies z^4 = -2 \; -2 = 2e^{i\pi}$. Roots are $z_k = \sqrt[4]{2}e^{i\frac{\pi+2k\pi}{4}}$ for $k = 0, 1, 2, 3$. $z_0 = \sqrt[4]{2}e^{i\pi/4}$, etc.

3. $(z + i)^n = (z - i)^n$ Since $z = i$ is not a solution, divide by $(z - i)^n$:

$$\left(\frac{z+i}{z-i}\right)^n = 1$$

Let $W = \frac{z+i}{z-i}$. Then W is an n-th root of unity. $W_k = e^{i\frac{2k\pi}{n}}$ for $k = 1, ..., n-1$ (k=0 gives $W = 1 \implies z + i = z - i \implies i = -i$ impossible). Solve for z:

$$z + i = W_k(z - i) \implies z(1 - W_k) = -i(1 + W_k)$$

$$z = -i\frac{1 + W_k}{1 - W_k} = -i\frac{1 + e^{i\theta_k}}{1 - e^{i\theta_k}}$$

Using half-angle formulas (similar to IV.1):

$$\frac{1 + e^{i\theta}}{1 - e^{i\theta}} = \frac{2\cos(\theta/2)e^{i\theta/2}}{-2i\sin(\theta/2)e^{i\theta/2}} = i\cot(\theta/2)$$

$$z_k = -i(i\cot\frac{k\pi}{n}) = \cot\frac{k\pi}{n}, \quad k = 1, ..., n-1$$

Solution to Exercise VI

Back to Exercise VI

Rigorous Solution

1. Verify identity

$$\frac{1+i\tan\alpha}{1-i\tan\alpha}=\frac{1+i\frac{\sin\alpha}{\cos\alpha}}{1-i\frac{\sin\alpha}{\cos\alpha}}=\frac{\cos\alpha+i\sin\alpha}{\cos\alpha-i\sin\alpha}=\frac{e^{i\alpha}}{e^{-i\alpha}}=e^{2i\alpha}$$

2. Show identity involving λ Let $\lambda=u+iv$. Since $|\lambda|=1, u^2+v^2=1$.

$$\frac{i(1-\lambda)}{1+\lambda}=i\frac{(1-\lambda)(1+\bar{\lambda})}{|1+\lambda|^2}=i\frac{1+\bar{\lambda}-\lambda-|\lambda|^2}{|1+\lambda|^2}$$

Since $|\lambda|^2=1$, numerator is $1+\bar{\lambda}-\lambda-1=\overline{\lambda}-\lambda=-2i\mathrm{Im}(\lambda)$.

$$\text{Term}=i\frac{-2i\mathrm{Im}(\lambda)}{|1+\lambda|^2}=\frac{2\mathrm{Im}(\lambda)}{(1+u)^2+v^2}$$

Denominator: $1+2u+u^2+v^2=2+2u=2(1+\mathrm{Re}(\lambda))$. Result: $\frac{2\mathrm{Im}(\lambda)}{2(1+\mathrm{Re}(\lambda))}=\frac{\mathrm{Im}(\lambda)}{1+\mathrm{Re}(\lambda)}$. □

3. Solve equation Equation: $(\frac{1+iz}{1-iz})^n=e^{2i\alpha}$. Let $W=\frac{1+iz}{1-iz}$. $W^n=e^{2i\alpha}$. Roots: $W_k=e^{i\frac{2\alpha+2k\pi}{n}}$. Solve for z: $1+iz=W_k(1-iz)\implies iz(1+W_k)=W_k-1$. $z=\frac{1}{i}\frac{W_k-1}{W_k+1}=-i\frac{e^{i\phi}-1}{e^{i\phi}+1}$. Using tangent half-angle identity: $\frac{e^{i\phi}-1}{e^{i\phi}+1}=i\tan(\phi/2)$. So $z=-i(i\tan(\phi/2))=\tan(\frac{2\alpha+2k\pi}{2n})=\tan(\frac{\alpha}{n}+\frac{k\pi}{n})$.

Solution to Exercise VII

Back to Exercise VII

Rigorous Solution

1. Sum $S_q=\sum_{k=0}^{n-1}(z_k)^q$ $z_k^q=e^{i\frac{2kq\pi}{n}}=(e^{i\frac{2q\pi}{n}})^k$. This is a geometric series with ratio $r=e^{i\frac{2q\pi}{n}}$. Case 1: If q is a multiple of n, $r=1$. Sum $=n\times1=n$. Case 2: If q is not a multiple of n, $r\neq1$.

$$S_q=\frac{1-r^n}{1-r}=\frac{1-e^{i2q\pi}}{1-r}=\frac{1-1}{1-r}=0$$

2. Prove $\sum_{k=0}^{n-1}(z+z_k)^n=n(z^n+1)$ Expand using Newton binomial:

$$\sum_{k=0}^{n-1}\sum_{j=0}^{n}C_n^jz^j(z_k)^{n-j}$$

Swap sums:

$$\sum_{j=0}^{n}C_n^jz^j\left(\sum_{k=0}^{n-1}(z_k)^{n-j}\right)$$

The inner sum is S_{n-j} from Part 1. S_{n-j} is non-zero only when exponent $n-j$ is a multiple of n. Since $0\leq j\leq n$, $0\leq n-j\leq n$. Possible multiples of n are 0 (when $j=n$) and n (when $j=0$). - For $j=n$: term is $C_n^nz^n(n)=nz^n$. - For $j=0$: term is $C_n^0z^0(n)=n$. Total Sum $=nz^n+n=n(z^n+1)$. □

3 Sequences of Real Numbers

3.1 Summary of Sequences

3.1.1 Definition and Basic Examples

Definition

A **sequence** is a function $a : \mathbb{N} \to \mathbb{R}$, denoted $(a_n)_{n=1}^{\infty}$ or simply (a_n). The number a_n is called the nth term.

Definition

A sequence (a_n) **converges** to a real number L if:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall n \geq N, |a_n - L| < \varepsilon$$

We write $\lim_{n\to\infty} a_n = L$ or $a_n \to L$.
If no such L exists, the sequence **diverges**.

Example

Basic Convergent Sequences:

1. Constant sequence: $a_n = c \to c$
2. $\lim_{n\to\infty} \frac{1}{n} = 0$
3. $\lim_{n\to\infty} \frac{1}{n^p} = 0$ for any $p > 0$
4. $\lim_{n\to\infty} \frac{n}{n+1} = 1$
5. $\lim_{n\to\infty} \sqrt[n]{a} = 1$ for $a > 0$

Example

Proof that $\lim_{n\to\infty} \frac{1}{n} = 0$**:**
Let $\varepsilon > 0$. By Archimedean property, choose $N > \frac{1}{\varepsilon}$. Then for all $n \geq N$:

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

3.1.2 Limit Theorems

Theorem

Limit Laws: Suppose $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$. Then:

1. $\lim_{n\to\infty}(a_n + b_n) = L + M$
2. $\lim_{n\to\infty}(ca_n) = cL$ for any constant $c \in \mathbb{R}$
3. $\lim_{n\to\infty}(a_n b_n) = LM$
4. If $M \neq 0$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$
5. If $a_n \leq b_n$ for all n, then $L \leq M$

Proof of (1). Given $\varepsilon > 0$, choose N_1 such that $|a_n - L| < \frac{\varepsilon}{2}$ for $n \geq N_1$, and N_2 such that $|b_n - M| < \frac{\varepsilon}{2}$ for $n \geq N_2$. Let $N = \max\{N_1, N_2\}$. Then for $n \geq N$:

$$|(a_n + b_n) - (L + M)| \leq |a_n - L| + |b_n - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

□

Theorem

Squeeze Theorem: If $a_n \leq b_n \leq c_n$ for all n, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Example

Example: Find $\lim_{n\to\infty} \frac{\sin n}{n}$.
Solution: Since $-1 \leq \sin n \leq 1$, we have $-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$. Since both $-\frac{1}{n}$ and $\frac{1}{n}$ converge to 0, by squeeze theorem, $\lim_{n\to\infty} \frac{\sin n}{n} = 0$.

3.1.3 Monotone Sequences

Definition

A sequence (a_n) is:

- **Increasing** if $a_n \leq a_{n+1}$ for all n
- **Decreasing** if $a_n \geq a_{n+1}$ for all n
- **Monotone** if it is either increasing or decreasing
- **Strictly monotone** if the inequalities are strict

Theorem

Monotone Convergence Theorem: Every bounded monotone sequence converges.

Proof. Suppose (a_n) is increasing and bounded above. Let $L = \sup\{a_n : n \in \mathbb{N}\}$. We show $a_n \to L$.

Given $\varepsilon > 0$, since $L - \varepsilon$ is not an upper bound, there exists N with $a_N > L - \varepsilon$. For $n \geq N$, since (a_n) is increasing:

$$L - \varepsilon < a_N \leq a_n \leq L < L + \varepsilon$$

Thus $|a_n - L| < \varepsilon$ for all $n \geq N$.

The decreasing case is similar with $L = \inf\{a_n\}$. □

Example

Example: The sequence $a_n = \left(1 + \frac{1}{n}\right)^n$ is increasing and bounded above by 3, hence converges to $e \approx 2.71828$.

3.1.4 Bounded Sequences

Definition

A sequence (a_n) is **bounded** if there exists $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.

Proposition 3.1. *Every convergent sequence is bounded.*

Proof. Suppose $a_n \to L$. Take $\varepsilon = 1$. There exists N such that $|a_n - L| < 1$ for all $n \geq N$. Then:

$$|a_n| \leq |a_n - L| + |L| < 1 + |L| \quad \text{for } n \geq N$$

Let $M = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, 1 + |L|\}$. Then $|a_n| \leq M$ for all n. □

Remark 3.2. *The converse is false: $a_n = (-1)^n$ is bounded but diverges.*

3.1.5 Subsequences

Definition

A **subsequence** of (a_n) is a sequence (a_{n_k}) where $n_1 < n_2 < n_3 < \cdots$ is an increasing sequence of indices.

Proposition 3.3. *If* $\lim_{n\to\infty} a_n = L$*, then every subsequence* (a_{n_k}) *also converges to* L*.*

Proof. Given $\varepsilon > 0$, there exists N such that $|a_n - L| < \varepsilon$ for all $n \geq N$. Since $n_k \to \infty$, there exists K such that $n_k \geq N$ for all $k \geq K$. Then for $k \geq K$, $|a_{n_k} - L| < \varepsilon$. □

Theorem

Bolzano-Weierstrass Theorem: Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Proof by Nested Intervals. Let (a_n) be bounded, say $a_n \in [c, d]$ for all n. Bisect $[c, d] = [c, m] \cup [m, d]$ where $m = \frac{c+d}{2}$. At least one half contains infinitely many terms of the sequence. Choose such a half and call it I_1. Pick n_1 with $a_{n_1} \in I_1$.

Bisect I_1 and choose a half I_2 containing infinitely many terms. Pick $n_2 > n_1$ with $a_{n_2} \in I_2$.

Continue inductively to get nested intervals I_k with length $\frac{d-c}{2^k} \to 0$, and indices $n_1 < n_2 < \cdots$ with $a_{n_k} \in I_k$.

By nested intervals theorem, $\bigcap_{k=1}^{\infty} I_k = \{L\}$. Since $a_{n_k} \in I_k$, we have $|a_{n_k} - L| \leq \frac{d-c}{2^k} \to 0$, so $a_{n_k} \to L$. □

3.1.6 Cauchy Sequences

Definition

A sequence (a_n) is **Cauchy** if:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall m, n \geq N, |a_n - a_m| < \varepsilon$$

Theorem

Cauchy Criterion: A sequence of real numbers converges if and only if it is a Cauchy sequence.

Proof. ($\Rightarrow$) If $a_n \to L$, given $\varepsilon > 0$, choose N such that $|a_n - L| < \frac{\varepsilon}{2}$ for all $n \geq N$. Then for $m, n \geq N$:

$$|a_n - a_m| \leq |a_n - L| + |L - a_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

($\Leftarrow$) Let (a_n) be Cauchy. First show it's bounded. Take $\varepsilon = 1$, there exists N such that $|a_n - a_N| < 1$ for all $n \geq N$. Then $|a_n| < |a_N| + 1$ for $n \geq N$, so bounded.

By Bolzano-Weierstrass, (a_n) has a convergent subsequence $a_{n_k} \to L$. We show $a_n \to L$.

Given $\varepsilon > 0$, choose N_1 such that $|a_n - a_m| < \frac{\varepsilon}{2}$ for $n, m \geq N_1$. Choose K such that $n_K \geq N_1$ and $|a_{n_k} - L| < \frac{\varepsilon}{2}$ for $k \geq K$. Then for $n \geq N_1$:

$$|a_n - L| \leq |a_n - a_{n_K}| + |a_{n_K} - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

□

3.1.7 Limit Superior and Limit Inferior

Definition

For a bounded sequence (a_n), define:

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \left(\sup_{k \geq n} a_k \right)$$

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \left(\inf_{k \geq n} a_k \right)$$

Proposition 3.4. *For a bounded sequence (a_n):*

1. $\liminf a_n \leq \limsup a_n$
2. $\lim a_n$ *exists iff* $\liminf a_n = \limsup a_n$, *and then* $\lim a_n = \liminf a_n = \limsup a_n$
3. $\limsup a_n$ *is the largest subsequential limit*
4. $\liminf a_n$ *is the smallest subsequential limit*

Example

Example: For $a_n = (-1)^n \left(1 + \frac{1}{n}\right)$:

$$\limsup a_n = \lim_{n\to\infty} \sup_{k \geq n} a_k = 1$$

$$\liminf a_n = \lim_{n\to\infty} \inf_{k \geq n} a_k = -1$$

Subsequential limits are -1 and 1.

3.2 List of Exercises

Exercise

Prove, by using the definition, that:

1. $\lim\limits_{n\to\infty} \dfrac{\ln(1+e^n)}{n} = 1$;
2. The variable $x_n = 3^{2n-1}$ is infinitely large as $n \to \infty$;
3. $\lim\limits_{n\to\infty} (1-2n) = -\infty$;
4. $\lim\limits_{n\to\infty} \dfrac{n^2-n+2}{3n^2+2n-4} = \dfrac{1}{3}$.

Go to Proof I

Exercise

Prove that if the sequence (u_n) is convergent, then $(|u_n|)$ is convergent. Is the opposite true? Explain your answer. Go to Proof II

Exercise

Prove that if (u_n) and (v_n) are convergent to l_1 and l_2, respectively, then:

$$\forall n \in \mathbb{N} : u_n \leq v_n \Rightarrow l_1 \leq l_2.$$

Go to Proof III

Exercise

Prove that the sequences (u_n) and (v_n) defined below are convergent.

1. $\forall n \in \mathbb{N} : u_n \leq \alpha, v_n \leq \beta$ and $u_n + v_n \to \alpha + \beta$;
2. $\forall n \in \mathbb{N} : 0 \leq u_n \leq 1, 0 \leq v_n \leq 1$ and $u_n v_n \to 1$.

Go to Proof IV

Exercise

Find the limit of the sequence (u_n), given by:

1. $u_n = \left(1 + \frac{x}{n}\right)^n, x \in \mathbb{R}$;
2. $u_n = \sum_{k=1}^{n-1} \frac{k}{n}$;
3. $u_n = \sum_{k=1}^{n} \frac{k!}{(2n)!}$;
4. $u_n = \sum_{k=0}^{n} (C_n^k)^{-1}$.

Go to Proof V

Exercise

Determine whether the sequence (u_n) defined as follows is convergent or divergent. If it is convergent, then find its limit.

1. $u_{n+1} = \frac{1}{u_n}, u_0 \neq 0$;
2. $u_{n+1} = 2u_n - 3, u_0 \geq 4$;
3. $u_{n+1} = 3 - \frac{4}{2+u_n}, u_0 = 4$.

Go to Proof VI

Exercise

We consider the sequence (u_n), defined by:

$$u_1 = 1, \quad \forall n \in \mathbb{N}(n \geq 2) : u_n = 1 + \frac{1}{u_{n-1}}.$$

1. Prove that the two subsequences (u_{2n}) and (u_{2n+1}) are adjacent.
2. Conclude that (u_n) is convergent.

Go to Proof VII

Exercise

Suppose (u_n) is a Cauchy sequence, with a subsequence (u_{n_k}) that converges to $l \in \mathbb{R}$. Prove that (u_n) converges to l. **Go to Proof VIII**

Exercise

Study, by using the Cauchy criterion, the nature of the sequence (u_n), defined by:

1. $u_n = \frac{1}{2} - \frac{1}{2^2+\ln 2} + \frac{1}{2^3+\ln 3} - \cdots + (-1)^{n+1}\frac{1}{2^n+\ln n}$;
2. $u_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$;
3. $u_n = \frac{1}{2+u_{n-1}}, u_0 > 0$.

Go to Proof IX

3.3 Solutions of Exercises

Solution to Exercise I

Back to Exercise I

Rigorous Solution

1. Prove $\lim_{n\to\infty} \dfrac{\ln(1+e^n)}{n} = 1$
Note: $\ln(1+e^n) = \ln(e^n(1+e^{-n})) = n + \ln(1+e^{-n})$. Then:

$$\frac{\ln(1+e^n)}{n} = 1 + \frac{\ln(1+e^{-n})}{n}.$$

Since $e^{-n} \to 0$, we have $\ln(1+e^{-n}) \to 0$. Thus the second term tends to 0. Therefore the limit is 1.
Rigorous proof using ϵ-definition: Given $\epsilon > 0$, we need N such that for $n > N$,

$$\left|\frac{\ln(1+e^n)}{n} - 1\right| < \epsilon.$$

This simplifies to $\frac{|\ln(1+e^{-n})|}{n} < \epsilon$. Since $0 < \ln(1+e^{-n}) < e^{-n}$ for large n, it suffices to have $\frac{e^{-n}}{n} < \epsilon$. This holds for sufficiently large n because e^{-n} decays faster than any polynomial. □
2. Prove $x_n = 3^{2n-1} \to +\infty$
Formal definition: For any $M > 0$, we need N such that for $n > N$, $3^{2n-1} > M$. Since $3^{2n-1} = \frac{1}{3}9^n$, and $9^n \to +\infty$, such N exists. □
3. Prove $\lim_{n\to\infty}(1-2n) = -\infty$
Definition: For any $M < 0$, we need N such that for $n > N$, $1-2n < M$. Solve $1-2n < M \Rightarrow n > \frac{1-M}{2}$. Choose $N = \lfloor \frac{1-M}{2} \rfloor + 1$. □
4. Prove $\lim_{n\to\infty} \dfrac{n^2-n+2}{3n^2+2n-4} = \dfrac{1}{3}$
Write:

$$\frac{n^2-n+2}{3n^2+2n-4} = \frac{1-\frac{1}{n}+\frac{2}{n^2}}{3+\frac{2}{n}-\frac{4}{n^2}}.$$

As $n \to \infty$, numerator $\to 1$ and denominator $\to 3$. Hence the limit is $\frac{1}{3}$. □

Solution to Exercise II

Back to Exercise II

Rigorous Solution

Claim: If (u_n) converges to l, then $(|u_n|)$ converges to $|l|$.
Proof: Using the reverse triangle inequality: $||u_n| - |l|| \leq |u_n - l|$. Since $|u_n - l| \to 0$, by the squeeze theorem, $||u_n| - |l|| \to 0$. Hence $|u_n| \to |l|$. □
Is the opposite true? No. Counterexample: $u_n = (-1)^n$. Then $|u_n| = 1$ converges to 1, but (u_n) does not converge. □

Solution to Exercise III

Back to Exercise III

Rigorous Solution

ssume $u_n \leq v_n$ for all n, and $u_n \to l_1$, $v_n \to l_2$. Suppose, for contradiction, that $l_1 > l_2$. Let $\epsilon = \frac{l_1 - l_2}{2} > 0$.
Since $u_n \to l_1$, there exists N_1 such that for $n > N_1$, $|u_n - l_1| < \epsilon$, i.e., $u_n > l_1 - \epsilon$.
Since $v_n \to l_2$, there exists N_2 such that for $n > N_2$, $|v_n - l_2| < \epsilon$, i.e., $v_n < l_2 + \epsilon$.
Let $N = \max(N_1, N_2)$. Then for $n > N$, we have:

$$u_n > l_1 - \epsilon = l_1 - \frac{l_1 - l_2}{2} = \frac{l_1 + l_2}{2},$$

$$v_n < l_2 + \epsilon = l_2 + \frac{l_1 - l_2}{2} = \frac{l_1 + l_2}{2}.$$

Hence $u_n > v_n$ for $n > N$, contradicting the hypothesis $u_n \leq v_n$. Therefore, $l_1 \leq l_2$. □

Solution to Exercise IV

Back to Exercise IV

Rigorous Solution

Part 1: Given $u_n \leq \alpha$, $v_n \leq \beta$, and $u_n + v_n \to \alpha + \beta$.
Since $u_n + v_n \to \alpha + \beta$, for any $\epsilon > 0$, there exists N such that for $n > N$, $|(u_n + v_n) - (\alpha + \beta)| < \epsilon$.
Suppose u_n does not converge to α. Then there exists $\epsilon_0 > 0$ and a subsequence u_{n_k} such that $u_{n_k} \leq \alpha - \epsilon_0$. Since $v_{n_k} \leq \beta$, we have:

$$u_{n_k} + v_{n_k} \leq \alpha - \epsilon_0 + \beta = \alpha + \beta - \epsilon_0.$$

This contradicts the convergence of $u_n + v_n$ to $\alpha + \beta$. Hence $u_n \to \alpha$. Similarly, $v_n \to \beta$. □
Part 2: Given $0 \leq u_n \leq 1$, $0 \leq v_n \leq 1$, and $u_n v_n \to 1$.
Since $u_n v_n \to 1$, both sequences are bounded. We show both converge to 1. Assume u_n does not converge to 1. Then there exists $\epsilon > 0$ and a subsequence u_{n_k} such that $u_{n_k} \leq 1 - \epsilon$. Then:

$$u_{n_k} v_{n_k} \leq (1 - \epsilon) \cdot 1 = 1 - \epsilon,$$

contradicting $u_n v_n \to 1$. Hence $u_n \to 1$. Similarly, $v_n \to 1$. □

Solution to Exercise V

Back to Exercise V

Rigorous Solution

1. Limit of $u_n = \left(1 + \frac{x}{n}\right)^n$
This is a classical result: $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$.

2. Limit of $u_n = \sum_{k=1}^{n-1} \frac{k}{n}$

$$u_n = \frac{1}{n} \sum_{k=1}^{n-1} k = \frac{1}{n} \cdot \frac{(n-1)n}{2} = \frac{n-1}{2} \to +\infty.$$

So $u_n \to +\infty$.

3. Limit of $u_n = \sum_{k=1}^{n} \frac{k!}{(2n)!}$

For fixed k, $\frac{k!}{(2n)!} \to 0$ as $n \to \infty$. However, we sum over k from 1 to n. Note that for large n, the largest term is when $k = n$: $\frac{n!}{(2n)!} \leq \frac{1}{(n+1)^n} \to 0$. Moreover, the sum is bounded by $n \cdot \frac{n!}{(2n)!} \to 0$. Hence $u_n \to 0$.

4. Limit of $u_n = \sum_{k=0}^{n} (C_n^k)^{-1}$
We compute:

$$u_n = \sum_{k=0}^{n} \frac{1}{C_n^k} = \sum_{k=0}^{n} \frac{k!(n-k)!}{n!}$$

By symmetry $C_n^k = C_n^{n-k}$, we have:

$$u_n = \frac{1}{C_n^0} + \frac{1}{C_n^1} + \cdots + \frac{1}{C_n^{n-1}} + \frac{1}{C_n^n} = 2 + \sum_{k=1}^{n-1} \frac{1}{C_n^k}$$

For $1 \leq k \leq n-1$, we have $C_n^k \geq n$, with equality only for $k = 1$ and $k = n-1$. Thus:

$$0 < \sum_{k=1}^{n-1} \frac{1}{C_n^k} \leq \sum_{k=1}^{n-1} \frac{1}{n} = \frac{n-1}{n} < 1$$

More precisely, for fixed k, as $n \to \infty$:

$$C_n^k = \frac{n!}{k!(n-k)!} \sim \frac{n^k}{k!} \to \infty$$

so $\frac{1}{C_n^k} \to 0$.
Therefore, by the squeeze theorem:

$$\lim_{n\to\infty} \sum_{k=1}^{n-1} \frac{1}{C_n^k} = 0$$

Hence:

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \left(2 + \sum_{k=1}^{n-1} \frac{1}{C_n^k}\right) = 2 + 0 = 2$$

Conclusion: $\lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{C_n^k} = 2.$ □

Solution to Exercise VI

Back to Exercise VI

Rigorous Solution

1. $u_{n+1} = \frac{1}{u_n}, u_0 \neq 0$
This gives $u_1 = \frac{1}{u_0}, u_2 = u_0, u_3 = \frac{1}{u_0}$, etc. So the sequence is periodic with period 2 unless $u_0 = \pm 1$. If $u_0 = 1$, then $u_n = 1$ for all n. If $u_0 = -1$, then $u_n = -1$ for all n. Otherwise, it oscillates and does not converge.
2. $u_{n+1} = 2u_n - 3, u_0 \geq 4$
This is a linear recurrence. Solve: $u_{n+1} - 3 = 2(u_n - 3)$. Let $v_n = u_n - 3$. Then $v_{n+1} = 2v_n$, so $v_n = 2^n v_0$. Thus $u_n = 3 + 2^n(u_0 - 3)$. Since $u_0 \geq 4$, $u_0 - 3 \geq 1$, so $u_n \to +\infty$.
3. $u_{n+1} = 3 - \frac{4}{2+u_n}, u_0 = 4$
Look for fixed point: $l = 3 - \frac{4}{2+l} \Rightarrow l^2 + 2l = 6l + 6 - 4 \Rightarrow l^2 - 4l + 2 = 0 \Rightarrow l = 2 \pm \sqrt{2}$.
Show monotonicity: Compute $u_1 = 3 - \frac{4}{6} = \frac{7}{3} \approx 2.333$. Since $u_0 > u_1$, check if decreasing. Assume $u_n > u_{n+1}$. Then:

$$u_{n+2} - u_{n+1} = \left(3 - \frac{4}{2 + u_{n+1}}\right) - \left(3 - \frac{4}{2 + u_n}\right) = 4\left(\frac{1}{2 + u_n} - \frac{1}{2 + u_{n+1}}\right).$$

Since $u_n > u_{n+1}$, the difference is positive, so $u_{n+2} > u_{n+1}$. Actually, the sequence oscillates. Better to study function $f(x) = 3 - \frac{4}{2+x}$. Its derivative is $f'(x) = \frac{4}{(2+x)^2} > 0$, so increasing. Since $f(x) < x$ for $x > 2 + \sqrt{2}$ and $f(x) > x$ for $x < 2 + \sqrt{2}$, and $u_0 = 4 > 2 + \sqrt{2}$, the sequence decreases and converges to $2 + \sqrt{2}$.

Solution to Exercise VII

Back to Exercise VII

Rigorous Solution

iven: $u_1 = 1$, $u_n = 1 + \frac{1}{u_{n-1}}$ for $n \geq 2$.
1. Show (u_{2n}) and (u_{2n+1}) are adjacent
First, compute: $u_2 = 2$, $u_3 = 1.5$, $u_4 \approx 1.6667$, $u_5 \approx 1.6$.
Claim: (u_{2n}) is decreasing, (u_{2n+1}) is increasing.
Proof by induction: Assume $u_{2k} > u_{2k+2}$ and $u_{2k+1} < u_{2k+3}$.
Check: $u_{2k+2} = 1 + \frac{1}{u_{2k+1}}$, $u_{2k+4} = 1 + \frac{1}{u_{2k+3}}$. Since $u_{2k+1} < u_{2k+3}$, we have $\frac{1}{u_{2k+1}} > \frac{1}{u_{2k+3}}$, so $u_{2k+2} > u_{2k+4}$. Similarly for odd terms.
Also, $u_{2n} - u_{2n+1} \to 0$ because the sequence is bounded and the recurrence forces the difference to shrink.
Hence they are adjacent.
2. Convergence
Adjacent sequences converge to the same limit l. Hence $u_n \to l$. Solving $l = 1 + \frac{1}{l}$ gives $l^2 - l - 1 = 0$, so $l = \frac{1+\sqrt{5}}{2}$ (the golden ratio). □

Solution to Exercise VIII

Back to Exercise VIII

Rigorous Solution

et (u_n) be Cauchy and $(u_{n_k}) \to l$. We want to show $u_n \to l$.
Let $\epsilon > 0$. Since (u_n) is Cauchy, there exists N_1 such that for all $m, n > N_1$, $|u_m - u_n| < \frac{\epsilon}{2}$.
Since $(u_{n_k}) \to l$, there exists K such that for all $k > K$, $|u_{n_k} - l| < \frac{\epsilon}{2}$.
Choose $N = \max(N_1, n_K)$. For any $n > N$, pick k such that $n_k > N$ (possible since $n_k \to \infty$).
Then:

$$|u_n - l| \leq |u_n - u_{n_k}| + |u_{n_k} - l| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $u_n \to l$. □

Solution to Exercise IX

Back to Exercise IX

Rigorous Solution

1. $u_n = \sum_{j=1}^{n} (-1)^{j+1} \frac{1}{2^j + \ln j}$
This is an alternating series. The terms $a_j = \frac{1}{2^j + \ln j}$ decrease to 0. By the alternating series test, the series converges. Hence the sequence of partial sums (u_n) converges.
2. $u_n = \sum_{k=1}^{n} \frac{1}{k}$ **(Harmonic series)**
The harmonic series diverges. To see using Cauchy criterion: For any N, choose $n = 2N$, $m = N$.
Then:

$$|u_n - u_m| = \sum_{k=N+1}^{2N} \frac{1}{k} \geq N \cdot \frac{1}{2N} = \frac{1}{2}.$$

So the Cauchy criterion fails. Hence (u_n) diverges.
3. $u_{n+1} = \frac{1}{2+u_n}, u_0 > 0$
Show it is Cauchy. First, find possible limit: $l = \frac{1}{2+l} \Rightarrow l^2 + 2l - 1 = 0 \Rightarrow l = -1 \pm \sqrt{2}$. Since $u_0 > 0$, $l = \sqrt{2} - 1$.
Show $|u_{n+1} - l| \leq \frac{1}{2}|u_n - l|$. Indeed:

$$u_{n+1} - l = \frac{1}{2+u_n} - \frac{1}{2+l} = \frac{l - u_n}{(2+u_n)(2+l)}.$$

Since $u_n > 0$, $(2+u_n)(2+l) > 2(2+\sqrt{2}-1) = 2(1+\sqrt{2}) > 2$. Hence $|u_{n+1} - l| \leq \frac{1}{2}|u_n - l|$.
Thus it converges exponentially, hence Cauchy. □

4 Real Functions of a Real Variable

4.1 Summary of Real Functions

4.1.1 Definition and Basic Concepts

Definition

A **function** f from a set A to a set B, denoted $f : A \to B$, is a rule that assigns to each element $x \in A$ exactly one element $f(x) \in B$.

Definition

Let $f : A \to B$ be a function.

1. The **domain** of f is A
2. The **codomain** is B
3. The **image** or range is $f(A) = \{f(x) : x \in A\} \subseteq B$
4. The **graph** of f is $\{(x, f(x)) : x \in A\} \subseteq A \times B$

4.1.2 Classification of Functions

Definition

A function $f : A \to B$ is:

1. **Injective (one-to-one)** if $f(x_1) = f(x_2) \implies x_1 = x_2$
2. **Surjective (onto)** if $f(A) = B$, i.e., $\forall y \in B, \exists x \in A$ with $f(x) = y$
3. **Bijective** if it is both injective and surjective

Example

Examples:

1. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is neither injective nor surjective
2. $f : [0, \infty) \to \mathbb{R}$, $f(x) = x^2$ is injective but not surjective
3. $f : \mathbb{R} \to [0, \infty)$, $f(x) = x^2$ is surjective but not injective
4. $f : [0, \infty) \to [0, \infty)$, $f(x) = x^2$ is bijective

4.1.3 Monotonicity

Definition

Let $f : I \to \mathbb{R}$ where $I \subseteq \mathbb{R}$ is an interval.

1. f is **increasing** if $x_1 < x_2 \implies f(x_1) \leq f(x_2)$
2. f is **strictly increasing** if $x_1 < x_2 \implies f(x_1) < f(x_2)$
3. f is **decreasing** if $x_1 < x_2 \implies f(x_1) \geq f(x_2)$
4. f is **strictly decreasing** if $x_1 < x_2 \implies f(x_1) > f(x_2)$
5. f is **monotone** if it is either increasing or decreasing

Proposition 4.1. *A strictly monotone function is injective.*

Proof. If f is strictly increasing and $x_1 \neq x_2$, say $x_1 < x_2$, then $f(x_1) < f(x_2)$, so $f(x_1) \neq f(x_2)$. □

4.1.4 Boundedness

Definition

Let $f : A \to \mathbb{R}$.

1. f is **bounded above** if $\exists M \in \mathbb{R}$ such that $f(x) \leq M$ for all $x \in A$
2. f is **bounded below** if $\exists m \in \mathbb{R}$ such that $f(x) \geq m$ for all $x \in A$
3. f is **bounded** if it is both bounded above and below

4.1.5 Even and Odd Functions

Definition

Let $f : \mathbb{R} \to \mathbb{R}$ (or $f : (-a, a) \to \mathbb{R}$).

1. f is **even** if $f(-x) = f(x)$ for all x
2. f is **odd** if $f(-x) = -f(x)$ for all x

Proposition 4.2. *1. Every function can be written uniquely as sum of an even and odd function:*

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}$$

2. The product of two even or two odd functions is even

3. The product of an even and odd function is odd

4.1.6 Composition of Functions

Definition

If $f : A \to B$ and $g : B \to C$, then the **composition** $g \circ f : A \to C$ is defined by $(g \circ f)(x) = g(f(x))$.

Proposition 4.3. *1. Composition is associative: $h \circ (g \circ f) = (h \circ g) \circ f$*

2. If f and g are injective, then $g \circ f$ is injective

3. If f and g are surjective, then $g \circ f$ is surjective

4. If f and g are bijective, then $g \circ f$ is bijective and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

4.1.7 Inverse Functions

Definition

If $f : A \to B$ is bijective, then the **inverse function** $f^{-1} : B \to A$ is defined by $f^{-1}(y) = x$ iff $f(x) = y$.

Theorem

A function $f : A \to B$ has an inverse $f^{-1} : B \to A$ if and only if f is bijective.

Example

Example: For $f : [0, \infty) \to [0, \infty)$, $f(x) = x^2$, the inverse is $f^{-1}(x) = \sqrt{x}$.

4.1.8 Asymptotics and Taylor Expansions

Definition

Asymptotics and Taylor

1. **Little-o notation:** $f(x) = o(g(x))$ as $x \to x_0$ if $\frac{f(x)}{g(x)} \to 0$.
2. **Taylor's theorem:** If f is n-times differentiable at x_0, then $f(x) = P_n(x) + o((x - x_0)^n)$, where P_n is the Taylor polynomial.
3. **Maclaurin series:** Taylor expansion at 0.

4.1.9 Limited Expansions (Nushour Mahdoudah)

Definition

Common Limited Expansions as $x \to 0$

1. **Exponential:** $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + o(x^n)$
2. **Logarithm:** $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1}\frac{x^n}{n} + o(x^n)$
3. **Trigonometric:**
   - $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + o(x^{2k+2})$
   - $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + o(x^{2k+1})$
   - $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + o(x^5)$
4. **Power functions:**
   - $(1 + x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + o(x^n)$
   - $\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + o(x^n)$
5. **Hyperbolic:**
   - $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2k+1}}{(2k+1)!} + o(x^{2k+2})$
   - $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2k}}{(2k)!} + o(x^{2k+1})$

4.1.10 Inverse Trigonometric and Hyperbolic Functions

Definition

Inverse Trigonometric Functions

1. **Definitions and domains:**
   - $\arcsin : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$
   - $\arccos : [-1, 1] \to [0, \pi]$
   - $\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$
2. **Fundamental identities:**
   - $\arcsin x + \arccos x = \frac{\pi}{2}$
   - $\arctan x + \arctan \frac{1}{x} = \frac{\pi}{2}$ for $x > 0$

- $\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right) + k\pi$ where k depends on quadrant

3. **Derivatives:**
   - $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$
   - $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$
   - $(\arctan x)' = \frac{1}{1+x^2}$

Definition

Hyperbolic and Inverse Hyperbolic Functions

1. **Definitions:**
   - $\sinh x = \frac{e^x - e^{-x}}{2}$
   - $\cosh x = \frac{e^x + e^{-x}}{2}$
   - $\tanh x = \frac{\sinh x}{\cosh x}$
2. **Inverse hyperbolic functions:**
   - $\operatorname{arsinh} x = \ln(x + \sqrt{x^2+1})$
   - $\operatorname{arcosh} x = \ln(x + \sqrt{x^2-1}), \quad x \geq 1$
   - $\operatorname{artanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \quad |x| < 1$
3. **Derivatives:**
   - $(\sinh x)' = \cosh x$
   - $(\cosh x)' = \sinh x$
   - $(\operatorname{arsinh} x)' = \frac{1}{\sqrt{x^2+1}}$
   - $(\operatorname{arcosh} x)' = \frac{1}{\sqrt{x^2-1}}$
   - $(\operatorname{artanh} x)' = \frac{1}{1-x^2}$
4. **Important identities:**
   - $\cosh^2 x - \sinh^2 x = 1$
   - $\sinh(2x) = 2\sinh x \cosh x$
   - $\cosh(2x) = \cosh^2 x + \sinh^2 x$

4.1.11 Function Comparison and Principal Parts

Definition

Comparison and Principal Parts

1. **Equivalence:** $f \sim g$ as $x \to x_0$ if $\lim_{x\to x_0} \frac{f(x)}{g(x)} = 1$
2. **Negligibility:** $f = o(g)$ as $x \to x_0$ if $\lim_{x\to x_0} \frac{f(x)}{g(x)} = 0$
3. **Order:** $f = O(g)$ as $x \to x_0$ if $\exists M > 0, \exists \delta > 0 : |f(x)| \leq M|g(x)|$ for $0 < |x - x_0| < \delta$
4. **Principal part:** For $f(x) \to 0$ as $x \to x_0$, the principal part is the simplest function $g(x)$ such that $f(x) \sim g(x)$

4.2 List of Exercises

Exercise

Prove, by using the definition, that:

1. $\lim\limits_{x\to+\infty} \dfrac{x+8}{x-1} = 1$;
2. $f(x) = 3^{2x-1}$ is infinitely large as $x \to +\infty$ and infinitely small as $x \to -\infty$;
3. $\lim\limits_{x\to+\infty} (1-2x) = -\infty$;
4. $\lim\limits_{x\to+\infty} \dfrac{x^2-x+2}{3x^2+2x-4} = \dfrac{1}{3}$;
5. $\lim\limits_{x\to1} \dfrac{3x+2}{2x-1} = 5$.

Go to Proof I

Exercise

Compute the limit of the function f as $x \to x_0 \in \overline{\mathbb{R}}$, in the following cases:

1. $f(x) = x - \sqrt{x^2-x}, \quad x_0 = +\infty$;
2. $f(x) = \left(1+\frac{1}{x}\right)^x, \quad x_0 = -\infty$;
3. $f(x) = \frac{\sqrt{2x^2+5x+9}}{x}, \quad x_0 = 0$;
4. $f(x) = \frac{\sqrt{x}-1}{\sqrt[3]{x}-1}, \quad x_0 = 1$.

Go to Proof II

Exercise

Show that the function f defined below does not have a limit as $x \to 0$.

1. $f(x) = \sin\frac{1}{x}$;
2. $f(x) = \cos\frac{\pi}{x}$.

Go to Proof III

Exercise

Compare the infinitesimals $\alpha(x)$ and $\beta(x) = x$ as $x \to 0$, and then find the order of $\alpha(x)$ with respect to $\beta(x)$, if possible, in the following cases:

1. $\alpha(x) = \sqrt{|\sin x|}$;
2. $\alpha(x) = x\cos\frac{1}{x}$;
3. $\alpha(x) = \sqrt[3]{x+1} - 1$.

Go to Proof IV

Exercise

Find the principal part of the infinitesimal (or infinite) $\alpha(x)$ as $x \to x_0$, in the following:

1. $\alpha(x) = x^3 - 3x + 2, \quad x_0 = 1;$
2. $\alpha(x) = \tan\left(x + \frac{\pi}{4}\right), \quad x_0 = \frac{\pi}{4}.$

Go to Proof V

Exercise

Prove that the function $f : x \mapsto \frac{1}{x}$ is continuous on $(0, 1]$. Is it uniformly continuous on $(0, 1]$?

Go to Proof VI

Exercise

Does the function $f : x \mapsto \sin(x + 1) \ln|x + 1|$ admit a continuous extension on $\mathbb{R}$?

Go to Proof VII

Exercise

Let $f, g : [0, 1] \to \mathbb{R}$ be two bounded functions and $h : \mathbb{R} \to \mathbb{R}$ be a function defined by:

$$\forall x \in \mathbb{R} : h(x) = \sup_{t \in [0,1]} (f(t) + xg(t)).$$

Show that h is a Lipschitz function. Go to Proof VIII

Exercise

Prove that:

1. $\forall x \in [0, 1] : \arctan \frac{\sqrt{1-x^2}}{x} = \arccos x;$
2. $\forall x \in [1, +\infty) : \operatorname{argh} \sqrt{\frac{1+x}{2}} = \frac{1}{2} \operatorname{argh} x;$
3. $\forall (a, b) \in \mathbb{R}^2, a < b : \frac{b-a}{1+b^2} < \arctan b - \arctan a < \frac{b-a}{1+a^2}.$

Go to Proof IX

Exercise

Solve in $\mathbb{R}$ the following equations:

1. $\arccos x = \arcsin 2x;$
2. $\arctan \frac{1}{x} + \arctan \frac{1}{x-3} + \arctan \frac{1}{x+3} = \frac{\pi}{4}.$

Go to Proof X

Exercise

Using the definition of the derivative, determine which of the following functions is differentiable at x_0.

1. $y = x^2 + |x+1|, \quad x_0 = -1;$
2. $y = x^2 + |x+1|, \quad x_0 = 1;$
3. $y = \frac{x}{1+e^{1/x}}, \quad y(0) = 0, \quad x_0 = 0.$

Go to Proof XI

Exercise

Compute the derivatives of the following functions:

1. $y = \arcsin \frac{x}{2};$
2. $y = \operatorname{argh} \frac{x^3+2x}{3x^2+1};$
3. $y = \frac{3t^2}{1+t^3}, \quad x = \frac{3t^3}{1+t^3}.$

Go to Proof XII

Exercise

Let f be a function defined by $f(x) = e^{x^2}\cos x$. Show that for all $\alpha > 0$, the equation $f'(x) = 0$ has at least one solution on $]-\alpha, \alpha[$. Go to Proof XIII

Exercise

Using the Mean Value Theorem, show that:

$$\forall x \in \mathbb{R}_+^* : \frac{1}{x+1} < \ln(1+x) - \ln x < \frac{1}{x}.$$

Deduce that:

$$\lim_{x\to+\infty} x(\ln(1+x) - \ln x) = 1 \quad \text{and} \quad \lim_{x\to+\infty} \left(1 + \frac{1}{x}\right)^x = e.$$

Go to Proof XIV

Exercise

Find a limited development of order n in a neighborhood of x_0 of the function f defined by:

1. $f(x) = (x^3+1)\sqrt{1-x}, \quad x_0 = 0, n = 3;$
2. $f(x) = \cos x \ln(1+x), \quad x_0 = 0, n = 4;$
3. $f(x) = \sin x \cos 2x, \quad x_0 = 0, n = 6;$
4. $f(x) = \frac{\sin x - 1}{\cos x + 1}, \quad x_0 = 0, n = 2;$
5. $f(x) = e^{\frac{x}{x+1}}, \quad x_0 = 0, n = 3;$
6. $f(x) = (\cos 2x)^2, \quad x_0 = \frac{\pi}{6}, n = 4.$

Go to Proof XV

Exercise

Using the MaclaurinLagrange formula, find an interval of values of the variable x for which the approximate formula

$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

is satisfied with a precision of up to 5×10^{-5}. Go to Proof XVI

4.3 Solutions of Exercises

Solution to Exercise I

Back to Exercise I

Rigorous Solution

1. $\lim_{x\to+\infty} \dfrac{x+8}{x-1} = 1$

For $x > 1$, $\frac{x+8}{x-1} = 1 + \frac{9}{x-1}$. Given $\epsilon > 0$, choose $X = 1 + \frac{9}{\epsilon}$. Then for $x > X$, $\left|\frac{9}{x-1}\right| < \epsilon$. Hence the limit is 1.

2. $f(x) = 3^{2x-1}$

As $x \to +\infty$, $2x-1 \to +\infty$, so $3^{2x-1} \to +\infty$. As $x \to -\infty$, $2x-1 \to -\infty$, so $3^{2x-1} \to 0$.

3. $\lim_{x\to+\infty} (1-2x) = -\infty$

For any $M < 0$, choose $X > \frac{1-M}{2}$. Then for $x > X$, $1 - 2x < M$.

4. $\lim_{x\to+\infty} \dfrac{x^2-x+2}{3x^2+2x-4} = \dfrac{1}{3}$

Divide numerator and denominator by x^2:

$$\frac{1-\frac{1}{x}+\frac{2}{x^2}}{3+\frac{2}{x}-\frac{4}{x^2}} \to \frac{1}{3}.$$

5. $\lim_{x\to1} \dfrac{3x+2}{2x-1} = 5$

For x near 1, $2x-1 \neq 0$. Direct substitution gives $\frac{5}{1} = 5$. For a rigorous ϵ-δ proof, note:

$$\left|\frac{3x+2}{2x-1} - 5\right| = \left|\frac{3x+2-10x+5}{2x-1}\right| = \left|\frac{-7x+7}{2x-1}\right| = 7\frac{|x-1|}{|2x-1|}.$$

For x close to 1, say $|x-1| < \frac{1}{4}$, then $|2x-1| > \frac{1}{2}$. Hence the expression is bounded by $14|x-1|$. Choose $\delta = \min\left(\frac{1}{4}, \frac{\epsilon}{14}\right)$. □

Solution to Exercise II

Back to Exercise II

Rigorous Solution

1. $f(x) = x - \sqrt{x^2-x}$ as $x \to +\infty$

Rationalize:

$$f(x) = \frac{(x-\sqrt{x^2-x})(x+\sqrt{x^2-x})}{x+\sqrt{x^2-x}} = \frac{x}{x+\sqrt{x^2-x}}.$$

As $x \to +\infty$, $\sqrt{x^2-x} = x\sqrt{1-\frac{1}{x}} = x\left(1-\frac{1}{2x}+o\left(\frac{1}{x}\right)\right)$. So denominator $\approx 2x - \frac{1}{2}$. Hence $f(x) \to \frac{1}{2}$.

2. $f(x) = \left(1+\frac{1}{x}\right)^x$ **as** $x \to -\infty$

Set $y = -x \to +\infty$. Then:

$$\left(1+\frac{1}{x}\right)^x = \left(1-\frac{1}{y}\right)^{-y} = \left(\frac{y}{y-1}\right)^y = \left(1+\frac{1}{y-1}\right)^y.$$

This tends to e.

3. $f(x) = \frac{\sqrt{2x^2+5x+9}}{x}$ **as** $x \to 0$

For $x \to 0^+$, $\sqrt{2x^2+5x+9} \to 3$, so $f(x) \to +\infty$. For $x \to 0^-$, $f(x) \to -\infty$. Hence the limit does not exist.

4. $f(x) = \frac{\sqrt{x}-1}{\sqrt[3]{x}-1}$ **as** $x \to 1$

Use the substitution $t^6 = x$. Then as $x \to 1$, $t \to 1$. The expression becomes:

$$\frac{t^3-1}{t^2-1} = \frac{(t-1)(t^2+t+1)}{(t-1)(t+1)} = \frac{t^2+t+1}{t+1} \to \frac{3}{2}.$$

So the limit is $\frac{3}{2}$. □

Solution to Exercise III

Back to Exercise III

Rigorous Solution

1. $f(x) = \sin\frac{1}{x}$
Consider the sequences $x_n = \frac{1}{2n\pi}$ and $y_n = \frac{1}{(2n+1/2)\pi}$. Then:

$$f(x_n) = \sin(2n\pi) = 0, \quad f(y_n) = \sin\left(2n\pi + \frac{\pi}{2}\right) = 1.$$

Both sequences tend to 0, but the limits of f are different. Hence the limit does not exist.
2. $f(x) = \cos\frac{\pi}{x}$
Consider $x_n = \frac{1}{n}$ and $y_n = \frac{2}{2n+1}$. Then:

$$f(x_n) = \cos(n\pi) = (-1)^n, \quad f(y_n) = \cos\left(\frac{(2n+1)\pi}{2}\right) = 0.$$

Again, different subsequential limits, so no limit. □

Solution to Exercise IV

Back to Exercise IV

Rigorous Solution

1. $\alpha(x) = \sqrt{|\sin x|}$
As $x \to 0$, $|\sin x| \sim |x|$. Hence $\alpha(x) \sim \sqrt{|x|}$. So $\alpha(x) = o(x)$ and the order is $\frac{1}{2}$.
2. $\alpha(x) = x\cos\frac{1}{x}$
Since $|\cos\frac{1}{x}| \leq 1$, we have $|\alpha(x)| \leq |x|$. But $\frac{\alpha(x)}{x} = \cos\frac{1}{x}$ has no limit. Hence $\alpha(x)$ is not comparable to x in the sense of order.
3. $\alpha(x) = \sqrt[3]{x+1} - 1$
Use the expansion: $(1+u)^{1/3} = 1 + \frac{u}{3} - \frac{u^2}{9} + o(u^2)$. With $u = x$:

$$\alpha(x) = 1 + \frac{x}{3} - \frac{x^2}{9} + o(x^2) - 1 = \frac{x}{3} + o(x).$$

Hence $\alpha(x) \sim \frac{x}{3}$, order 1. □

Solution to Exercise V

Back to Exercise V

Rigorous Solution

1. $\alpha(x) = x^3 - 3x + 2$ at $x_0 = 1$
Note that $\alpha(1) = 0$. Factor: $x^3 - 3x + 2 = (x-1)(x^2 + x - 2) = (x-1)^2(x+2)$. As $x \to 1$, $x + 2 \to 3$. Hence the principal part is $3(x-1)^2$.
2. $\alpha(x) = \tan\left(x + \frac{\pi}{4}\right)$ **at** $x_0 = \frac{\pi}{4}$
Let $t = x - \frac{\pi}{4}$. Then $x + \frac{\pi}{4} = \frac{\pi}{2} + t$. So:

$$\tan\left(\frac{\pi}{2} + t\right) = -\cot t = -\frac{1}{t} + \frac{t}{3} + o(t).$$

Hence the principal part is $-\frac{1}{t} = -\frac{1}{x - \frac{\pi}{4}}$. □

Solution to Exercise VI

Back to Exercise VI

Rigorous Solution

Continuity on $(0, 1]$
The function $f(x) = \frac{1}{x}$ is a rational function, continuous wherever the denominator is nonzero. On $(0, 1]$, $x \neq 0$, so f is continuous.
Uniform continuity?
Check the definition: For $\epsilon = 1$, we need δ such that for all $x, y \in (0, 1]$, $|x - y| < \delta \implies \left|\frac{1}{x} - \frac{1}{y}\right| < 1$.
Take $x_n = \frac{1}{n}$, $y_n = \frac{1}{n+1}$. Then $|x_n - y_n| = \frac{1}{n(n+1)} \to 0$, but:

$$\left|\frac{1}{x_n} - \frac{1}{y_n}\right| = |n - (n+1)| = 1.$$

So for any δ, we can choose n large enough such that $|x_n - y_n| < \delta$ but the difference is 1. Hence not uniformly continuous. □

Solution to Exercise VII

Back to Exercise VII

Rigorous Solution

he function $f(x) = \sin(x+1)\ln|x+1|$ is defined for $x \neq -1$. The only problematic point is $x = -1$.
As $x \to -1$, set $t = x + 1 \to 0$. Then:

$$f(x) = \sin t \cdot \ln|t|.$$

Since $\sin t \sim t$ and $\ln|t| \to -\infty$, the product tends to $0 \cdot (-\infty)$, which is indeterminate. More precisely:

$$|\sin t \ln|t|| \leq |t||\ln|t|| \to 0.$$

So $\lim_{x \to -1} f(x) = 0$. Hence we can define $f(-1) = 0$ to obtain a continuous extension on $\mathbb{R}$. □

Solution to Exercise VIII

Back to Exercise VIII

Rigorous Solution

et $M_f = \sup_{t\in[0,1]} |f(t)|$, $M_g = \sup_{t\in[0,1]} |g(t)|$ (finite since f, g bounded).
For any $x, y \in \mathbb{R}$:

$$|h(x) - h(y)| = \left|\sup_t (f(t) + xg(t)) - \sup_t (f(t) + yg(t))\right| \leq \sup_t |(x-y)g(t)| = |x-y| \cdot M_g.$$

The inequality follows from the property $|\sup A - \sup B| \leq \sup |A - B|$.
Hence h is Lipschitz with constant M_g. □

Solution to Exercise IX

Back to Exercise IX

Rigorous Solution

1. Identity: $\arctan \frac{\sqrt{1-x^2}}{x} = \arccos x$ for $x \in [0, 1]$
Let $\theta = \arccos x$. Then $x = \cos\theta$, $\sqrt{1-x^2} = \sin\theta$. So $\frac{\sqrt{1-x^2}}{x} = \tan\theta$. Since $\theta \in [0, \frac{\pi}{2}]$, $\arctan(\tan\theta) = \theta$. Hence the equality.
2. Identity involving hyperbolic functions
Let $t = \operatorname{argh} x$. Then $x = \cosh t$. Compute:

$$\sqrt{\frac{1+x}{2}} = \sqrt{\frac{1+\cosh t}{2}} = \cosh\frac{t}{2}.$$

Hence $\operatorname{argh}\left(\cosh\frac{t}{2}\right) = \frac{t}{2} = \frac{1}{2}\operatorname{argh} x$.
3. Inequality for arctan
Apply the Mean Value Theorem to $\arctan x$ on $[a, b]$: There exists $c \in (a, b)$ such that:

$$\arctan b - \arctan a = (b-a) \cdot \frac{1}{1+c^2}.$$

Since $\frac{1}{1+b^2} < \frac{1}{1+c^2} < \frac{1}{1+a^2}$, the inequality follows. □

Solution to Exercise X

Back to Exercise X

Rigorous Solution

1. $\arccos x = \arcsin 2x$
Domain: $x \in [-1, 1]$ and $2x \in [-1, 1] \Rightarrow x \in [-\frac{1}{2}, \frac{1}{2}]$.
Take cosine of both sides: $x = \cos(\arcsin 2x) = \sqrt{1-4x^2}$.
Squaring: $x^2 = 1 - 4x^2 \Rightarrow 5x^2 = 1 \Rightarrow x = \pm\frac{1}{\sqrt{5}}$.
Check: For $x = \frac{1}{\sqrt{5}}$, $\arccos\frac{1}{\sqrt{5}} > 0$ and $\arcsin\frac{2}{\sqrt{5}} > 0$, both in $(0, \frac{\pi}{2})$, valid.
For $x = -\frac{1}{\sqrt{5}}$, $\arccos(-\frac{1}{\sqrt{5}}) \in (\frac{\pi}{2}, \pi)$ while $\arcsin(-\frac{2}{\sqrt{5}}) \in (-\frac{\pi}{2}, 0)$, not equal. Hence only solution: $x = \frac{1}{\sqrt{5}}$.
2. $\arctan\frac{1}{x} + \arctan\frac{1}{x-3} + \arctan\frac{1}{x+3} = \frac{\pi}{4}$
Use the formula $\arctan a + \arctan b = \arctan\frac{a+b}{1-ab} \pmod{\pi}$.
First combine first two:

$$\arctan\frac{1}{x} + \arctan\frac{1}{x-3} = \arctan\frac{\frac{1}{x} + \frac{1}{x-3}}{1 - \frac{1}{x(x-3)}} = \arctan\frac{2x-3}{x^2-3x-1}.$$

Then add the third:

$$\arctan\frac{2x-3}{x^2-3x-1}+\arctan\frac{1}{x+3}=\arctan\frac{\frac{2x-3}{x^2-3x-1}+\frac{1}{x+3}}{1-\frac{2x-3}{(x^2-3x-1)(x+3)}}.$$

Simplify the fraction to get $\frac{3x^2-9}{x^3-9x-9}$. Set equal to $\tan\frac{\pi}{4}=1$:

$$\frac{3x^2-9}{x^3-9x-9}=1\Rightarrow x^3-9x-9=3x^2-9\Rightarrow x^3-3x^2-9x=0.$$

Factor: $x(x^2-3x-9)=0\Rightarrow x=0$ or $x=\frac{3\pm\sqrt{9+36}}{2}=\frac{3\pm3\sqrt{5}}{2}$.
Check each for validity (ensuring arguments are defined and the sum equals $\frac{\pi}{4}$). All are valid. □

Solution to Exercise XI

Back to Exercise XI

Rigorous Solution

1. $y=x^2+|x+1|$ at $x_0=-1$
Near -1, $x+1$ changes sign. For $x<-1$, $|x+1|=-(x+1)$. For $x>-1$, $|x+1|=x+1$.
Left derivative: $f'_-(-1)=\lim_{h\to0^-}\frac{(-1+h)^2-(-1+h+1)-[(-1)^2-(-1+1)]}{h}=\lim_{h\to0^-}\frac{1-2h+h^2+h-1}{h}=\lim_{h\to0^-}(-1+h)=-1$.
Right derivative: $f'_+(-1)=\lim_{h\to0^+}\frac{(-1+h)^2+(-1+h+1)-1}{h}=\lim_{h\to0^+}\frac{1-2h+h^2+h-1}{h}=\lim_{h\to0^+}(-1+h)=-1$.
Both equal, so differentiable at -1.
2. At $x_0=1$
For $x>1$, $|x+1|=x+1$. The function is $y=x^2+x+1$, derivative $2x+1$, so $f'(1)=3$.
Actually, near 1, $|x+1|=x+1$ always, so differentiable.
3. $y=\frac{x}{1+e^{1/x}}$ at $x_0=0$, with $y(0)=0$
Check limit of difference quotient:

$$\lim_{x\to0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to0}\frac{1}{1+e^{1/x}}.$$

This limit does not exist because as $x\to0^+$, $e^{1/x}\to+\infty$, so limit 0; as $x\to0^-$, $e^{1/x}\to0$, so limit 1. Hence not differentiable. □

Solution to Exercise XII

Back to Exercise XII

Rigorous Solution

1. $y=\arcsin\frac{x}{2}$
Derivative: $y'=\frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\cdot\frac{1}{2}=\frac{1}{\sqrt{4-x^2}}$.
2. $y=\operatorname{argh}\frac{x^3+2x}{3x^2+1}$
Use formula: $(\operatorname{argh}u)'=\frac{u'}{\sqrt{u^2-1}}$ for $|u|>1$.
Compute $u=\frac{x^3+2x}{3x^2+1}$. Then:

$$u'=\frac{(3x^2+2)(3x^2+1)-(x^3+2x)(6x)}{(3x^2+1)^2}=\frac{9x^4+3x^2+6x^2+2-6x^4-12x^2}{(3x^2+1)^2}=\frac{3x^4-3x^2+2}{(3x^2+1)^2}.$$

Then $y'=\frac{u'}{\sqrt{u^2-1}}$.
3. Parametric: $y=\frac{3t^2}{1+t^3},x=\frac{3t^3}{1+t^3}$

Compute $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$.
$dy/dt = \frac{6t(1+t^3)-3t^2\cdot 3t^2}{(1+t^3)^2} = \frac{6t+6t^4-9t^4}{(1+t^3)^2} = \frac{6t-3t^4}{(1+t^3)^2} = \frac{3t(2-t^3)}{(1+t^3)^2}$.
$dx/dt = \frac{9t^2(1+t^3)-3t^3\cdot 3t^2}{(1+t^3)^2} = \frac{9t^2+9t^5-9t^5}{(1+t^3)^2} = \frac{9t^2}{(1+t^3)^2}$.
Hence $\frac{dy}{dx} = \frac{3t(2-t^3)}{9t^2} = \frac{2-t^3}{3t}$. □

Solution to Exercise XIII

Back to Exercise XIII

Rigorous Solution

unction: $f(x) = e^{x^2}\cos x$.
Compute $f'(x) = 2xe^{x^2}\cos x - e^{x^2}\sin x = e^{x^2}(2x\cos x - \sin x)$.
We want to show $f'(x) = 0$ has a solution in $(-\alpha, \alpha)$ for any $\alpha > 0$.
Note f is continuous and differentiable. Apply Rolle's theorem on $[-\alpha, \alpha]$.
Compute $f(-\alpha) = e^{\alpha^2}\cos(-\alpha) = e^{\alpha^2}\cos\alpha$.
$f(\alpha) = e^{\alpha^2}\cos\alpha$.
So $f(-\alpha) = f(\alpha)$. By Rolle's theorem, there exists $c \in (-\alpha, \alpha)$ such that $f'(c) = 0$. □

Solution to Exercise XIV

Back to Exercise XIV

Rigorous Solution

pply the Mean Value Theorem to $g(t) = \ln t$ on $[x, x+1]$: There exists $c \in (x, x+1)$ such that:

$$\ln(x+1) - \ln x = \frac{1}{c}.$$

Since $x < c < x+1$, we have $\frac{1}{x+1} < \frac{1}{c} < \frac{1}{x}$, which gives the inequality.
Deduction of limits:
Multiply the inequality by x:

$$\frac{x}{x+1} < x(\ln(x+1) - \ln x) < 1.$$

As $x \to +\infty$, the left side tends to 1, so by squeeze theorem, the middle term tends to 1.
For the second limit, note:

$$x(\ln(x+1) - \ln x) = \ln\left(1 + \frac{1}{x}\right)^x.$$

Hence $\left(1+\frac{1}{x}\right)^x \to e$. □

Solution to Exercise XV

Back to Exercise XV

Rigorous Solution

se known expansions:
$\sqrt{1-x} = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} + o(x^3)$.
$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + o(x^4)$.
$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)$.

$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)$.
$\cos 2x = 1 - 2x^2 + \frac{2}{3}x^4 + o(x^4)$.
$e^u = 1 + u + \frac{u^2}{2} + \frac{u^3}{6} + o(u^3)$.
1. $f(x) = (x^3+1)\sqrt{1-x}$
Multiply: $(1+x^3)(1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} + o(x^3)) = 1 - \frac{x}{2} - \frac{x^2}{8} + \left(-\frac{1}{16} + 1\right)x^3 + o(x^3) = 1 - \frac{x}{2} - \frac{x^2}{8} + \frac{15}{16}x^3 + o(x^3)$.
2. $f(x) = \cos x \ln(1+x)$
Multiply expansions: $\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + o(x^4)\right)$.
Compute up to x^4: $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^3}{2} + \frac{x^4}{4} + \frac{x^5}{24} + \dots$ Actually, careful collection:

- x
- $-\frac{x^2}{2}$
- $\frac{x^3}{3} - \frac{x^3}{2} = -\frac{x^3}{6}$
- $-\frac{x^4}{4} + \frac{x^4}{4} + \frac{x^4}{24} = \frac{x^4}{24}$

So $f(x) = x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + o(x^4)$.
3. $f(x) = \sin x \cos 2x$
Multiply: $\left(x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)\right)\left(1 - 2x^2 + \frac{2}{3}x^4 + o(x^4)\right)$.
Up to x^5: $x - 2x^3 + \frac{2}{3}x^5 - \frac{x^3}{6} + \frac{x^5}{3} + \frac{x^5}{120} + o(x^5) = x - \frac{13}{6}x^3 + \left(\frac{2}{3} + \frac{1}{3} + \frac{1}{120}\right)x^5 = x - \frac{13}{6}x^3 + \frac{121}{120}x^5 + o(x^5)$.
4. $f(x) = \frac{\sin x - 1}{\cos x + 1}$
As $x \to 0$, numerator $\sim -1 + x$, denominator $\sim 2 - \frac{x^2}{2}$. So:

$$f(x) = \frac{-1+x}{2 - \frac{x^2}{2}} = \left(-\frac{1}{2} + \frac{x}{2}\right)\left(1 + \frac{x^2}{4} + o(x^2)\right) = -\frac{1}{2} + \frac{x}{2} - \frac{x^2}{8} + o(x^2).$$

5. $f(x) = e^{\frac{x}{x+1}}$
Write $\frac{x}{x+1} = 1 - \frac{1}{x+1} = 1 - (1 - x + x^2 - x^3 + o(x^3)) = x - x^2 + x^3 + o(x^3)$.
Then $e^u = 1 + (x - x^2 + x^3) + \frac{1}{2}(x - x^2)^2 + \frac{1}{6}x^3 + o(x^3) = 1 + x - x^2 + x^3 + \frac{1}{2}(x^2 - 2x^3) + \frac{1}{6}x^3 + o(x^3) = 1 + x - \frac{1}{2}x^2 + \frac{2}{3}x^3 + o(x^3)$.
6. $f(x) = (\cos 2x)^2$ **at** $x_0 = \frac{\pi}{6}$
Let $t = x - \frac{\pi}{6}$. Then $\cos 2x = \cos\left(\frac{\pi}{3} + 2t\right) = \frac{1}{2}\cos 2t - \frac{\sqrt{3}}{2}\sin 2t$.
Square and expand in powers of t. Up to t^4:

$$\cos 2t = 1 - 2t^2 + \frac{2}{3}t^4 + o(t^4), \quad \sin 2t = 2t - \frac{4}{3}t^3 + o(t^4).$$

So $\cos 2x = \frac{1}{2}(1 - 2t^2 + \frac{2}{3}t^4) - \frac{\sqrt{3}}{2}(2t - \frac{4}{3}t^3) + o(t^4) = \frac{1}{2} - \sqrt{3}t - t^2 + \frac{2\sqrt{3}}{3}t^3 + \frac{1}{3}t^4 + o(t^4)$.
Square this expression up to t^4. The constant term is $\frac{1}{4}$. The linear term is $2 \cdot \frac{1}{2} \cdot (-\sqrt{3}t) = -\sqrt{3}t$.
Quadratic: from linear squared: $3t^2$, from constant times quadratic: $2 \cdot \frac{1}{2} \cdot (-t^2) = -t^2$, total $2t^2$. Cubic: from linear times quadratic: $2 \cdot (-\sqrt{3}t)(-t^2) = 2\sqrt{3}t^3$, from constant times cubic: $2 \cdot \frac{1}{2} \cdot \frac{2\sqrt{3}}{3}t^3 = \frac{2\sqrt{3}}{3}t^3$, total $\frac{8\sqrt{3}}{3}t^3$. Quartic: from quadratic squared: t^4, from linear times cubic: $2 \cdot (-\sqrt{3}t) \cdot \frac{2\sqrt{3}}{3}t^3 = -4t^4$, from constant times quartic: $2 \cdot \frac{1}{2} \cdot \frac{1}{3}t^4 = \frac{1}{3}t^4$, total $-\frac{8}{3}t^4$.
So $f(x) = \frac{1}{4} - \sqrt{3}t + 2t^2 + \frac{8\sqrt{3}}{3}t^3 - \frac{8}{3}t^4 + o(t^4)$. □

Solution to Exercise XVI

Back to Exercise XVI

Rigorous Solution

he MaclaurinLagrange formula for $\cos x$ with remainder in Lagrange form:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{\sin \xi}{6!} x^6,$$

for some ξ between 0 and x.
The error term is $R_5(x) = -\frac{\sin \xi}{720} x^6$. Since $|\sin \xi| \leq 1$, we have:

$$|R_5(x)| \leq \frac{|x|^6}{720}.$$

We want this $\leq 5 \times 10^{-5}$. Solve:

$$\frac{|x|^6}{720} \leq 5 \times 10^{-5} \Rightarrow |x|^6 \leq 0.036 \Rightarrow |x| \leq (0.036)^{1/6}.$$

Compute: $0.036^{1/6} = e^{\frac{1}{6} \ln 0.036}$. $\ln 0.036 \approx -3.324$. Divide by 6: -0.554. Exponentiate: $e^{-0.554} \approx 0.574$.
So for $|x| \leq 0.574$, the approximation is within the desired precision. □

5 Conclusion and Resources

Conclusion

This textbook has systematically developed the core concepts of **Analysis I**, in accordance with the academic standards expected in first-year university mathematics. The material presented emphasizes not only computational proficiency, but also the rigorous logical framework that underpins mathematical analysis.

The exposition began with a thorough study of the real number system, highlighting its algebraic structure, order properties, and, most importantly, the completeness property. These fundamental characteristics were shown to play a central role in the formulation and justification of key results throughout the course. In particular, completeness emerged as a foundational principle upon which much of real analysis is constructed.

The study of sequences and their convergence introduced students to limiting processes and infinite behavior, treated with precise definitions and strict logical reasoning. Through detailed examples and exercises, essential notions such as boundedness, monotonicity, subsequences, and convergence were explored, alongside fundamental results such as the Bolzano–Weierstrass Theorem.

The chapter on functions provided a structured overview of function classifications and elementary function families, establishing a conceptual framework that will be further expanded in subsequent courses. Throughout the text, special attention was given to proof techniques, reinforcing the importance of clarity, logical coherence, and rigorous argumentation as indispensable tools in mathematical reasoning.

The exercises included in this textbook were carefully designed to progress from elementary applications to more advanced problems requiring abstraction and synthesis. This gradual increase in difficulty aims to develop both technical skill and mathematical maturity.

Mastery of the topics covered in this textbook provides a solid foundation for further study in advanced courses such as **Analysis II**, where continuity, differentiability, and integration are treated in greater depth. Moreover, it prepares students for more advanced fields including **Real Analysis**, **Complex Analysis**, and **Functional Analysis**.

In conclusion, this textbook is intended not merely as an introduction to analysis, but as a rigorous initiation into mathematical thinking, equipping students with the conceptual tools and logical discipline necessary for success in higher-level mathematics.

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Web Resources

- Wolfram MathWorld
- Practise Mathematics with Benbachir
- blackpenredpen
- Algeria National Higher School of Mathematics
- Moodle platform of Echahid Cheikh Larbi Tebessi University.

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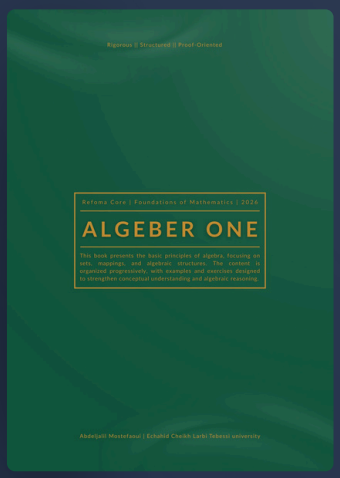
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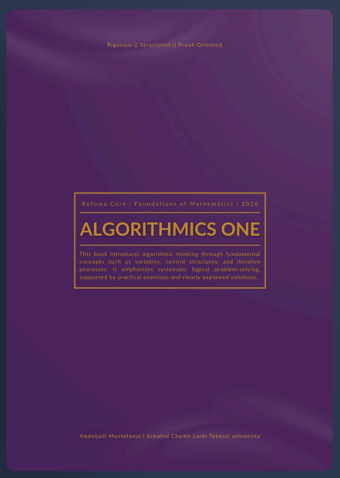
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